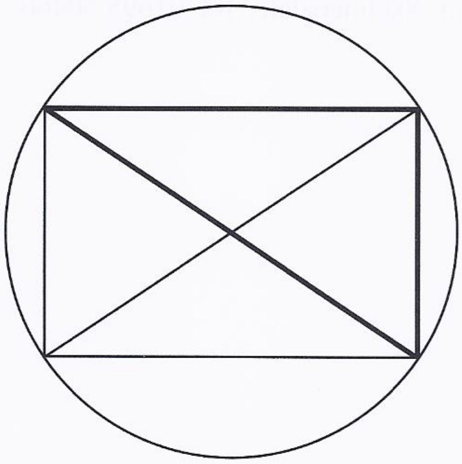
GEOMETRY
FOR AN ORAL TRADITION



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TO NEW BEGINNINGS

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Thanks are also due to all those whose comments have helped in any way, especially Kenneth Williams and James Armstrong.

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INTRODUCTION

How this book arose

The inspiration behind this book was two-fold. First, Tirthaji's reconstruction of vedic mathematics, of which a brief historical account follows later in the Introduction. Being an oral tradition, little or no ancient material is extant, and the one surviving book by Tirthaji contains only a handful of examples in geometry.

Secondly, I wondered if the elementary properties of a circle can be demonstrated simply, in such way that we can see why they hold good. Having satisfied myself that they can, the next issue was, to investigate what prior steps, such as definitions and axioms, would serve to establish these demonstrations as part of a system of geometry suitable for an oral tradition.

An oral tradition in geometry

The reader might like to consider, what might such a system be like?
As for me, an image comes to mind of an exposition being given in a sheltered nook on a beach, figures being sketched in the sand, and the rest of the exposition being spoken. Proofs would generally need to be brief and to the point. Qualities such as effortlessness, simplicity, brevity and clarity would be highly prized. The aim of this book is to provide a text suitable for such an exposition.

Two other aspects of an oral tradition are worthy of mention. First, the use of verse as an *aide memoire.* It is much easier to memorize material in rhyming verses – but this idea has not been used here. Secondly, there is the use of sutras, as in the vedic tradition, a sutra being a terse statement of some important point or principle (literally, a sutra is a thread). The material of this book was developed without reference to Tirthaji's sutras, but their application to this system is investigated in Appendix 1.

Anyone who has read through the first three of the thirteen books of Euclid's 'Elements' will have encountered the theorems on circles given here. That the present material covers the ground more swiftly is partly because less ground is covered, partly because these methods are generally much simpler and briefer.

B.K. Tirthaji's reconstruction of vedic mathematics

Ancient India's oral vedic tradition began to be written down about 1600 or 1700 B.C., according to western scholars. Over a period of about 1000 years the four vedas were written down: the Rig-veda, the Yajur-veda, the Sama-veda, and the Atharva-veda.

Tradition had it that the vedas were the embodiment of all knowledge. Yet when nineteenth century scholars examined the vedas there were some puzzles. Consider the Atharva-veda, for example, which deals with architecture, engineering, mathematics, and other topics. The material supposed to be on mathematics comes under the heading of 'Ganita sutras', i.e. mathematics sutras. Under this heading came statements such as, "In the reign of King Kamsa, arson, famine and insanitary conditions prevailed". The scholars could make nothing of it: there appeared to be no connection with mathematics.

However, a brilliant south Indian scholar, later known as Shri Bharati Krishna Tirthaji, was convinced that there was something in the ancient tradition. By persistence he obtained a clue (he tells us), and after that things began falling into place. In due course he concluded that the whole of mathematics, pure and applied, in all its branches, comes under sixteen sutras. He wrote sixteen volumes on the subject, which subsequently were all lost.

Tirthaji was born in 1884. His key work on vedic mathematics appears to have been done between the years 1911 and 1918. In 1921 he was made Shankaracharya of Puri (Hindu India being led by four Shankaracharyas, a bit like having four popes). Shortly before this he became a renunciate, i.e. he renounced his former life. This, and his considerable religious duties as Shankaracharya, are no doubt the reasons why he did not turn his attention to vedic mathematics again until the 1950s, only to realise that the sixteen volumes were lost. He decided to rewrite them all, and as a preliminary step wrote another book, Vedic Mathematics, to introduce the whole series. Owing to ill-health he got no further, and died in 1960. His introductory book, the only one by him surviving on the subject, was published in 1965.

A further issue

At the outset mathematics divides into two branches, based on number and form: arithmetic (from which stems algebra) and geometry. Tirthaji's introductory book deals mainly with arithmetic and algebra: geometry is scarcely addressed. Furthermore, the handful of examples he gives on geometry are unlike anything here. This material is something new. The present study does not simply arise out of his book. Yet it does use a mental approach. Can it be considered to belong to Tirthaji's system? If it does it complements his introductory material on vedic mathematics.

Geometry and the nature of an oral tradition

Imagine a society with an oral tradition, and willing to allow its understanding of subjects such as geometry to develop; willing to incorporate fresh insights into the tradition. It would be in their interests to do so, and it would happen quite naturally through teachers mastering the current understanding, and in some cases developing it. Such a society would probably have a fairly pragmatic outlook, having respect for the tradition but not regarding its current version as a perfect system, necessarily faultless, but rather as reflecting the current understanding, and as such subject to amendment, be it correction or further refinement. Perhaps this is in the nature of an oral tradition in geometry. Certainly the writing of the present book has been a bit like that, fresh insights constantly changing the material and the format, and it would be no surprise if it could usefully benefit from further insights and amendments, etc.

A word about the Preliminaries

An earlier draft of this book began in much the same way as Euclid's *Elements.* Subsequently it became clear that an earlier starting point was needed. The normal thing is to begin at the beginning, and then go on to the end. (Of course this is mathematics, so perhaps all normality is suspended!) But what kind of activity is it that begins at the beginning and then goes **backwards?** Somewhere Bertrand Russell says that it is the philosophy of mathematics, adding that once established it becomes mathematics.

The Preliminaries outline the new starting point, and the reasons for it.

PRELIMINARIES

" 'Tis evident that all the sciences have a relation, greater or less, to human nature; and that however wide any of them may seem to run from it, they still return back by one passage or another. Even **Mathematics**, **Natural Philosophy**, and **Natural Religion**, are in some measure dependent on the science of **Man**; since they lie under the cognizance of men, and are judged of by their powers and faculties."

David Hume, *A Treatise of Human Nature*.

The relevant 'faculties and powers' of man

Geometry is a branch of mathematics, which is a branch of human knowledge. Need there be any surprize, then, if it found useful to begin by stating the relevant faculties and abilities that we have? They are as follows:

1. LANGUAGE Since the concern here is with an oral tradition, the use of language is of paramount importance. Implicit in this are speech, listening and understanding what is said.

2. VISUAL MESSAGES (a) An oral tradition may be characterized by the lack of writing, but it would doubtless have a strong visual element: e.g. keen observation of something being shown or done, and an interest in such things as paintings and drawings and diagrams.
(b) Use of instruments is another relevant characteristic – especially their skilled use. Effectively, many instruments are an extension of the human body: they widen the range of what we can do. This study uses *drawing instruments and materials,* including a straight edge and compasses. On being used according to precise instructions, they ensure that figures are drawn accurately.

3. REASON Geometry is especially valuable as an exercise in the use of reason. For the present study it suffices to note that we have the ability to recognize sound reasoning.
One way or another, then, the following are available at the outset of the study:

1. A language (in use).
2. Drawing instruments and material. More specifically: a plane (usually represented by a sheet of paper), a pen, a straight edge and a pair of compasses. (The ability to use them is implied, for otherwise they are not truly available.)
3. The ability to recognize valid reasoning.

These three are called *provisions*, being things provided. They are what a suitably equipped human being brings to the study. What does the subject of geometry itself contribute?

A FURTHER PROVISION

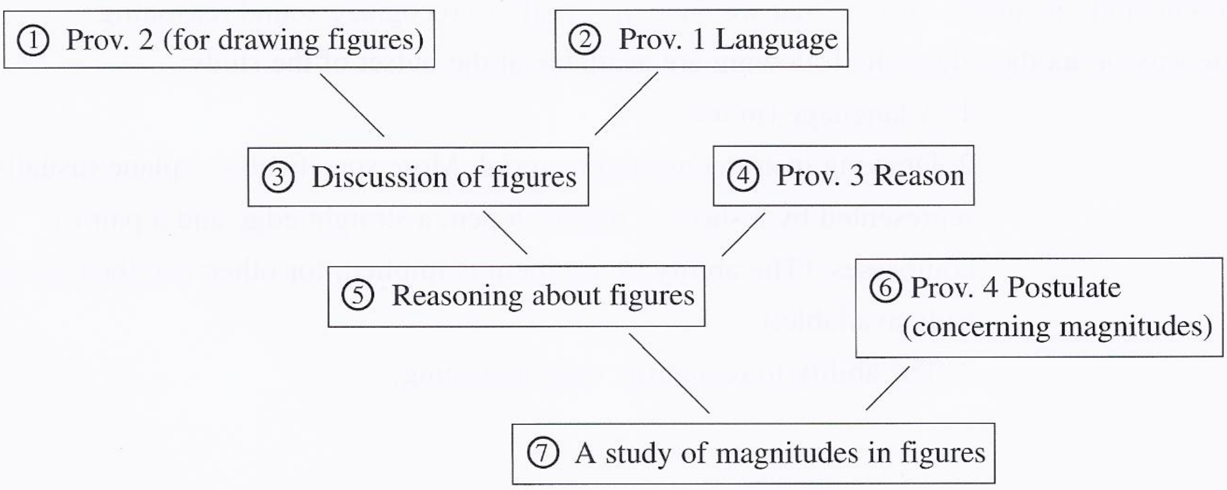
To the above list is added an assumption (a postulate), that *magnitudes are unchanged by motion*. The magnitudes of geometry are such things as *length, angle, area* and *volume*. It is the movement of each magnitude as a whole which is being referred to here. Since the present study of geometry is a *study of magnitudes in figures*, it is not surprising that something needs to be given initially concerning magnitudes, to start the ball rolling.

THE ROLES OF THE FOUR PROVISIONS

The study starts then with these four Provisions. But are they really essential? Why choose this starting point? A book on a craft commonly begins by discussing the tools and materials needed, and it is clearly appropriate that it should do so. In geometry this translates into starting with the relevant instruments and material, which is where the first three Provisions come in:

(i) *Language* is an instrument for the communication of thought.
(ii) *Drawing instruments and materials* are used to produce figures.
(iii) *The ability to recognize valid reasoning* is an instrument for evaluating the reasoning presented to us.

Each of the Provisions has a role, including the Postulate, as is shown in the following diagram.



PRELIMINARIES

Explanation of the diagram

Box 1: Granted Provision 2 (Box 1) we can draw figures.
3: And granted also a language, we can discuss them.
5: Granted, additionally, the ability to recognize valid reasoning, we can reason about figures.
7: Finally, by bringing in the fourth Provision it becomes possible to study the magnitudes in figures.

These descriptions refer to the functions of the Provisions, rather than a sequence in time. Having explained why provisions are needed, a few words about language follow.

Without the provision of a language there would be no oral tradition, no tool with which to carry out (or at least to communicate) the study. Yet in the case of geometry this language also provides material to work on. It contains words which imply a prior acquaintance with geometry. A word such as 'circle' brings a concept to mind. Thus in beginning the study of geometry there is already a considerable pool of knowledge to draw on, contained within the language*. This can be accessed through definitions.

Here we have a key device of language. *A definition names something and in some way, characterizes it uniquely*. A handy way of doing this, in mathematics, is to state a single property which is so distinctive as to suffice of itself – known as a defining property. This can then be drawn upon in proofs.

A principle used here is that *dictionary definitions apply until replaced*. This provides a justification for the procedure of examining a definition proposed in the text, to see if it is satisfactory. A definition proposed for a totally unknown term cannot be tested in the same way.

One of the functions of definitions is to remind the student, or else introduce the concept, but *which words need to be placed on the list of definitions?*

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* This has implications. As an example consider the terms: surface, plane surface, straight line and sphere. Knowing them enables us to state such things as that the surface of a sphere does not contain any straight lines, whereas a plane surface does.

PRELIMINARIES

The criterion used is that words in common use need not be defined*, but other words necessary to the study do need to be. That is to say, words in common use form a core of material available at the outset, and we are unlikely to choose to redefine any of them unless there is some specific advantage in doing so.

This criterion is selected as being a common-sense one, natural to an oral tradition.

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Note that alongside the text there occur explanations and comments under the general heading 'Remark(s)'. Not being referred to in proofs these may range freely forwards in the text as well as backwards. They are intended to aid understanding of the text. There is also a Commentary at the end of the book, which deals with a number of other points arising, but not needed to follow to follow the text. Its primary purpose is to deal with some objections that can be raised against this approach, the aim being to show that the system has a sound basis, with a coherent underlying viewpoint.

The Provisions now follow. They, along with the Definitions, provide a base upon which the Propositions rest.

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* Perhaps we should say *redefined*, for dictionary definitions are already available.

PROVISIONS

One way or another, the following are provided:

(1) A language.

(2) A plane, a pen, a straight edge and a pair of compasses.

(3) The ability to recognize valid reasoning.

(4) *Postulate:* Magnitudes are unchanged by motion

Note that Provision 2 is not considered to be effective until after the Definitions. This is so that the definitions of a plane and a pen etc. are given **before** they are granted.

Remarks concerning the Postulate

(1) There have been objections to the use of movement in geometry both in ancient times and more recently. These and other points are discussed and answered in the Commentary. The present system firmly and formally incorporates movement through this postulate.

(2) The Postulate implies that:

(a) lengths may be directly transferred anywhere in the plane (compasses at a fixed radius being one means of doing so);

(b) an angle can be rotated about its vertex (or fulcrum) without change;

(c) an angle can be translated in space without change (translation = movement without rotation);

(d) whether an angle is measured by rotation (of a radius), or by taking the difference between the two bounding directions, the result is the same.

Implications concerning area and other magnitudes need not concern us at present.

(3) The Postulate is relevant to some of the definitions (e.g. *line* and *angle*).

DEFINITIONS

THE GEOMETER'S DEVICES

Definition 1 An *axiom* is an assertion which is granted, being self-evident.

Definition 2 A *postulate* is an assertion which is granted, being an assumption.

Definition 3 A *provision* is something provided.
Remark Provisions include assertions, such as axioms and postulates, but are wider in scope.

Definition 4 A *theorem* is an assertion which is to be proved.

Definition 5 A *corollary* is a theorem which immediately follows from the theorem preceding it.
Remark A Corollary may be presented informally, without restatement of the details of its parent theorem, such as the figure.

Definition 6 A *problem*, or problem of construction, states a construction which is required, needing to be given and proved.

MAGNITUDE

Definition 7 *Magnitude* is that of which a part is indistinguishable from the whole save in size and position.
Remark Examples include lengths and angles. Magnitude is a key concept, because this study of geometry is *a study of magnitudes in figures*.

Definition 8 The whole of a magnitude is said to be *greater than* a part, and the latter to be *less than* the whole.

Definition 9 Magnitudes which are indistinguishable from one another except by position are said to be *equal* to one another.

DEFINITIONS

POINTS

Definition 10 A *point* has position but no parts.

LINES

Definition 11 The path traced out by a point on a moving body is called a *line*.

Remarks (1) The tip of a pen simulates a point; hence this definition, in accordance with what a pen traces out.

(2) It would be briefer to speak of a 'moving point' rather than a 'point on a moving body'. But as Bertrand Russell points out, the objection to this is that a point, being a position, cannot move. However, bodies can move, and there are points on them.

Definition 12 If two lines cross one another at a single point they are said to *intersect* at that point.

Remarks (1) Two intersecting lines are useful for specifying a point.

(2) It remains to be established that 'intersection' is possible, i.e., that two lines can cross one another at a single point. The result is intuitively obvious, and the proof reflects this.

[See A7 Corollary (2)].

Definition 13 That line which is uniquely specified given two points on it is said to be *straight*.

Remarks (1) Since a specific two points are not referred to it is implied that any two points on the straight line suffice.

(2) Defined in this way, a straight line is unlimited in either direction.

(3) Those two points can be used to name that straight line.

Definition 14 A straight line which is terminated at a point is said to *stand* on that point or, to be *drawn to* or *from* it.

Definition 15 That part of a straight line which suffices to connect two points on it is said to lie *between* those two points and is called a *length* or *segment*.

Remarks (1) And those two points are used to name that segment.

(2) Although either term may be used, if we wish to draw attention to the size of the segment then the term *length* is more suitable (I suggest).*
(3) The term *distance* is used as an alternative to length, especially if the segment is not actually drawn. An example is the radius of a circle.

Definition 16 *Direction* is specified by a straight line or segment drawn to or from a point.
Remark It is understood that 'holding constant direction' is the same thing as 'in a straight line'.

Definition 17 The direction from a point is said to be *opposite* to the direction to a point.
Notation For convenience, when considering the directions specified by a length or straight line AB, let us use AB to denote the direction from A to B and BA to denote the opposite direction.

Remark When dealing with lengths, direction is irrelevant. Thus lengths AB and BA are equal. Otherwise put, there are two ways of naming the same length.

TURNS

Definition 18 A *turn* is a change in direction.

Definition 19 If a length, initially coinciding with a straight line, turns just sufficiently about one of its ends to lie on the adjoining part of the straight line, it is said to make a *half turn*.

Definition 20 That turn is called a *complete* which suffices to return to the original direction, having passed through the opposite direction and another pair of opposite directions through the fulcrum.
Remark The second pair of opposite directions is included to ensure that the initial half turn is not then repeated in reverse.

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* This allows ambiguity, but only such as is to be found in a dictionary definition. Usually the context makes the meaning clear, as is normal in speech. If there were contexts where the ambiguity mattered, then of course steps would need to be taken to avoid it.

DEFINITIONS

SURFACES

Remark The definition of a surface comes next. The purpose of this particular definition is not to tell us what a surface is – those who want that would do better to look in a dictionary. Definition 21 incorporates two properties that a surface needs to have. Since we are being granted a plane surface, by including those two properties in the definition they are automatically granted, so saving us the trouble of proving them.

A more detailed account of this is given in the Commentary, but none of it is needed in order to understand the mathematics that follows. For that, a dictionary definition will do.

Definition 21 A *surface* is that which suffices to contain a complete turn at each of its points, and a multitude of lines linking any two of them.

Definition 22 That surface which is uniquely specified given any three non-collinear points on it is said to be a *plane*.

Remarks (1) This definition is an extension of the definition of a straight line – a point used to advantage in Proposition A9, a theorem which might otherwise appear as an axiom.

(2) Note that the definition appeals to experience. It is a formulation of something already known. Of course, the same remark applies to many another definition.

ANGLES

Definition 23 An *angle* is a measure of change in direction at a point.

Definition 24 A pair of straight lines or segments standing on a common point are said to *bound* the angle so formed.

Remarks (1) A half turn and a complete turn are examples of angles, the latter being bounded by coincident segments or straight lines.

(2) If an angle includes one or more complete turns, the number of such turns needs to be noted. Just noting the bounds of an angle is insufficient otherwise. However, this issue does not arise in the present book.

Definition 25 And that common point is called the *centre*, or *vertex*, or *fulcrum* of the angle.

DEFINITIONS

Definition 26 If a straight line standing on another straight line makes equal angles with it on either side then that first line is said to be *perpendicular* to the other, and those equal angles are said to be *right angles.*

Definition 27 If two angles share a common centre, and lie on either side of a common boundary, the angle contained within their outer boundaries is called their *sum* or *addition.*

FIGURES

Definition 28 A drawing in a plane is called a *figure.*

Remarks (1) Otherwise put, a figure consists of lines and/or points in a plane.

(2) A part of a figure is itself a figure, being a drawing in a plane.

Definition 29 A construction of a figure, or *construction*, consists of the steps required to draw that figure.

Definition 30 Figures which are indistinguishable from one another except by position are said to be *congruent* to one another.

Definition 31 If two straight lines in a plane do not meet they are said to be *parallel.*

Definition 32 If a pair of straight lines are separately intersected by a third straight line, the latter is said to be a *transversal* to the other two.

Remark Note that the pair of straight lines intersected by a transversal may or may not be parallel. This accords with the original use of the word 'transversal'. [See The Shorter Oxford Dictionary.]

Definition 33 Two similar parts of the same figure or of two different figures are said to *correspond* to one another.

Remark For example if two triangles are similar (Definition 45), the equal angles can be said to correspond to one another. Again, the angles similarly made by a transversal with two straight lines are said to be corresponding angles. And in similar triangles, the pair of sides sandwiched between the same pair of angles are said to be corresponding sides.

DEFINITIONS

THE CIRCLE

Definition 34 A *circle* consists of a line at constant distance from a point, and without end points.
Remark The word *distance* is used as an alternative to *length* here. *Constant distance* specifically refers to a measure.

Definition 35 And the line so drawn is called the *circumference* of the circle.

Definition 36 A part of the circumference is called an *arc*, or *arc of the circle*.

Definition 37 The given point is called the *centre* and the fixed distance from the centre is called the *radius* of the circle.

Definition 38 A segment lying between two points on the circumference of a circle is called a *chord* of the circle.

Definition 39 A chord passing through the centre of a circle is called a *diameter* of that circle.

Definition 40 A *semicircle* consists of a diameter of a circle and that part of the circle which lies on one side of it.

Definition 41 A straight line which shares just one point with a circle is said to be *a tangent* to that circle.

Definition 42 Joining the ends of an arc of a circle to a point on the opposite arc, the arc is said to *subtend* an angle at that point, or to subtend an angle on the circumference.

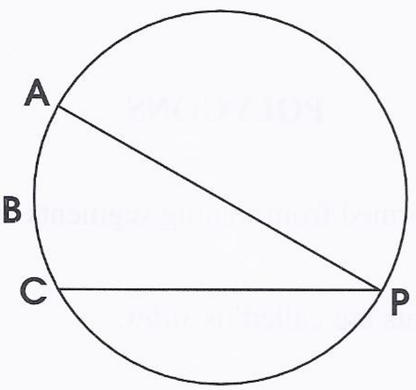


Fig 1 Showing an arc ABC subtending an angle at P

A chord can also be said to subtend an angle at the circumference, although there may be a need to specify which of the two opposite arcs the angle is subtended onto.

Definition 43 If two chords of a circle intersect inside it they are said to *intersect internally*.

Definition 44 And if the two chords do not intersect until extended outside the circle they are said to *intersect externally*.
Remark Internal and external intersection are illustrated below. In both cases chord AC is said to have parts AB and BC. These, it emerges, are sides of similar triangles.
(See e.g. Propositions D7 and D9).

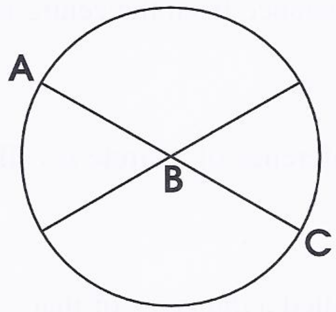


Fig 2a Internal intersection

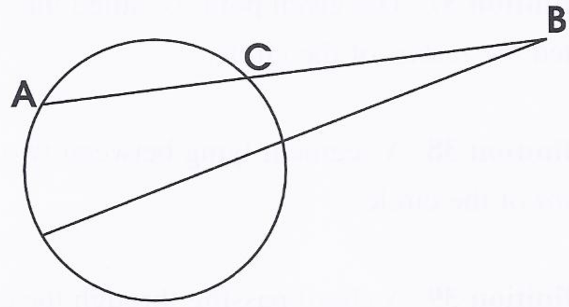


Fig 2b External intersection

Definition 45 A *one-line drawing* is one which can be drawn without removing pen from paper.

Definition 46 A *closed figure* is that part of a one-line drawing which suffices to meet itself.
Remarks (1) Of course, if the one-line drawing does not meet itself the figure is not closed.
(2) A circle is an example of a closed figure.

POLYGONS

Definition 47 A closed figure formed from joining segments end-to-end is called a *polygon*.

Definition 48 And those segments are called its *sides*.

Definition 49 Where two sides meet is called a *vertex* of the polygon.

DEFINITIONS

Definition 50 An angle within a polygon and between two adjacent sides is called *an angle of the polygon.*

Definition 51 The angle between one side extended and the adjacent side is called an *exterior* or *external angle* of the polygon.

Remarks (1) It is customary to use the term 'external angle' in this way.

(2) Note that there are two external angles at a vertex (Figure 3); being vertically opposite they are equal to one another, as will be shown.

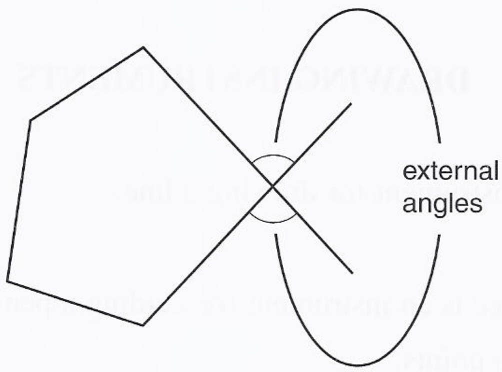


Fig 3

Definition 52 A triangle is a polygon with three sides.

Notation A triangle formed from joining three points, A, B and C, is called 'triangle ABC', and is denoted by ΔABC.

Definition 53 An *isosceles* triangle is one with two sides equal in length.

Definition 54 If two angles of one triangle are the same as two angles of another those two triangles are said to be *similar*.

Convention If two triangles are stated to be either similar or congruent to one another it is understood that, if the vertices are named, they are named in the same sequence.

Definition 55 A *quadrilateral* is a polygon with four sides.

Definition 56 A *rectangle* is a quadrilateral with all angles equal.

DEFINITIONS

Definition 57 A *square* is a rectangle with two adjoining sides equal in length.

Definition 58 A *cyclic-quadrilateral* is a quadrilateral with all four vertices lying on the circumference of a circle.
Remark Strictly speaking it is a theorem that a four-sided figure has four vertices, but it is assumed here.

Definition 59 A *parallelogram* is a quadrilateral with opposite sides parallel.

DRAWING INSTRUMENTS

Definition 60 A *pen* is an instrument for drawing a line.

Definition 61 A *straight edge* is an instrument for guiding a pen along any chosen part of the straight line specified by two points.
Remark A straight edge can be used to extend a segment.

Definition 62 A *pair of compasses* is an instrument for guiding a pen around the circumference of a circle, given the centre and the radius.

Remarks (1) This last instrument is called a 'pair' because it has two linked arms, the tip of one acting as a 'fixed point' and the tip of the other as a point on a moving body.

(2) Euclid does not permit the use of compasses to draw a circle of any given radius about any centre. The issue is discussed briefly in the remarks following Provision 4, these being followed up in Part IV of the Commentary.

Remark A further definition is given at the beginning of Part A and another at the beginning of Part C, placed there for the convenience of the reader. But generally it is more convenient to have all the definitions in one place.

DEFINITIONS

SOME POINTS CONCERNING THE TWO TYPES OF PROPOSITION

This study of geometry is concerned with:

(1) constructing figures (problems), and

(2) making assertions about figures, especially the magnitudes they contain (theorems).

These are the two types of proposition. They are interwoven in a single sequence, and need to work from a base of some given material – the Provisions. Of these:

(i) Provision 2 starts the constructions off, initially conceding circles (using compasses), and lengths (using a straight edge), and points (being granted a plane).

(ii) Provision 4 is an initial granted assertion.

(iii) In due course it becomes clear that this is not all, for Provision 1 provides additional material in the form of words. [see Commentary, Part II (f)]

One might ask, which are the more important, theorems or problems? The object of the present book is to demonstrate the elementary properties of circles, and so the text leads up to and concludes with the relevant theorems. The problems are subservient to this end.

PROPOSITIONS

PART A

CONGRUENCE, MAGNITUDES AND LINES

CONGRUENCE

Definition 63 A *specific* construction is one which produces only one figure.

Remark The following theorem applies to figures which can be drawn with a straight edge and compasses.

Proposition A1 Theorem Figures which are or can be constructed identically are congruent.

Proof: Two figures can use the same lengths, angles and other magnitudes. [Postulate]
Therefore a given construction can be used in two different places.
But if it is specific, a given construction produces only one figure. [Definition 63]
Therefore two figures so constructed are in fact the same.
This establishes the theorem.

MAGNITUDES

Proposition A2 Theorem Magnitudes which are or can be constructed identically are equal.

For that part of a figure needed to specify a magnitude is itself a figure. [Definition 28]

And if two such parts are or can be constructed identically they are congruent to one another. [A1 Theorem]

Therefore the magnitudes they specify are indistinguishable; i.e. they are equal to one another. [Definition 9]

This establishes the Theorem.

PROPOSITIONS

Proposition A3 ***Theorem*** Magnitudes which cannot be constructed identically are not equal.

For if two magnitudes cannot be constructed identically they are not indistinguishable, one from the other. Therefore they are not equal.

Remark Two abbreviated forms of reference may be used. For example, Proposition A9 Theorem may be referred to as Proposition A9 or A9 Theorem.

Proposition A4 ***Theorem*** Magnitudes which are equal to the same magnitude are equal to one another.

For (applying Definition 9) if two magnitudes are each indistinguishable from a third magnitude they are indistinguishable from one another.

Notation Denote the addition, or taking of magnitudes together by the + sign, and the converse, i.e. subtraction, by the – sign.

Proposition A5 ***Theorem*** If equals are added to equals the totals are equal.

Proof: Let a_1, a_2, b_1 and b_2 be magnitudes such that

$$a_1 = a_2$$
$$\& \; b_1 = b_2$$

Then a_1 is indistinguishable from a_2 and b_1 from b_2 [Definition 9]
Hence $a_1 + b_1$ is indistinguishable from $a_2 + b_2$
i.e. $a_1 + b_1 = a_2 + b_2$, demonstrating the Theorem.

Proposition A6 ***Theorem*** If equals are subtracted from equals the resultants are equal.

For let a_1, a_2, b_1 and b_2 be magnitudes such that

$$a_1 + b_1 = a_2 + b_2$$
$$\& \; b_1 = b_2$$

Then if $a_1 = a_2$ the previous theorem holds good, but if $a_1 \neq a_2$ it does not.
i.e. $a_1 = a_2$, demonstrating the Theorem.

PROPOSITIONS

***Proposition A7* Theorem** A point has no magnitude, for a point has no parts. [Definition 10]

And whatever lacks parts lacks the defining characteristic of a magnitude. [Definition 7]
That is, a point has no magnitude.

LINES

***Proposition A8* Theorem** A line is one point thick throughout.

This follows from a line being the path of a point on a moving body. [Definition11]

***A8* Corollary** If two lines cross one another, and do not share a common stretch of line, they cross at a single point.

***Proposition A9* Theorem** A line can be used to join two points.

For any two points on a line are joined by that line.

***A9* Corollary** A line has two end-points, unless they coincide.
For we start the line at one point and end at another, unless they coincide.

***Proposition A10* Theorem** Two distinct straight lines cannot share more than one point.

For two points suffice to specify a straight line. [Definition 13]

***Proposition A11* Theorem** A length has the property of magnitude.

For a part in no way differs from the whole save in size and position.

PROPOSITIONS

Proposition A12 ***Theorem*** The straight line through any two points in a plane lies in that plane.

For any two of the points which suffice to specify a plane also specify a straight line. [Definition 13]
Therefore that straight line lies in the plane.
But *any* three points on it suffice to specify a plane. [Definition 22]
The result follows.

Remark It can be argued that the next three theorems (the third a corollary) are so obvious that no proof is needed. Indeed the reader may safely ignore the proofs and move on – thereby treating them as axioms, which they could be.

Remark A circle divides the plane into three regions: points inside the circle, points outside it, and points on the circumference. This is a matter of description, just as the parts of figures are a matter of description.

Proposition A13 ***Theorem*** If two circles intersect one another they do so twice.

For if two circles intersect, part of one lies inside the other and part lies outside. That is, that circle has one arc inside the other and one arc outside.
And each arc has two end points. [*A9 Corollary*]
Since the two arcs meet, being complementary parts of the same circle, they must share the same end points.
And these end points are neither inside nor outside the circle. That is, they lie on the circumference.
Therefore if two circles intersect they do so twice.

Proposition A14 ***Theorem*** A chord of a circle lies inside that circle.

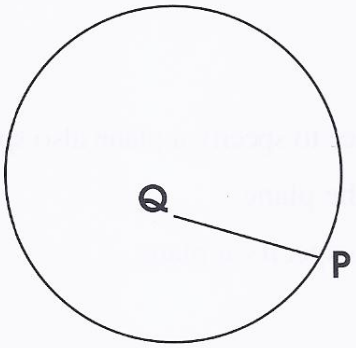


Fig 4

Take a point P on the circle and join it to a point Q inside the circle.
Then segment PQ lies inside the circle (Figure 4).
Extend PQ in the direction PQ. Let the extension be to a point R.
There are three possibilities as to the location of R:

(i) inside the circle (Figure 5 (i))
(ii) outside the circle (Figure 5 (ii))
(iii) on the circumference (Figure 5 (iii)).

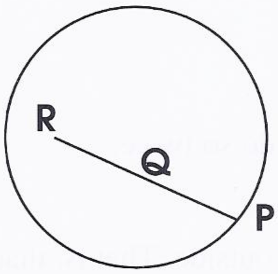


Fig 5(i)

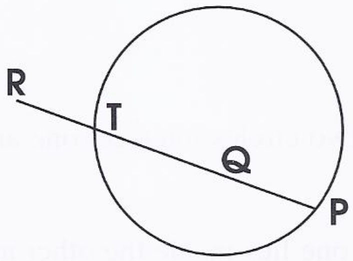


Fig 5(ii)

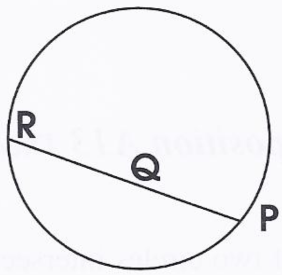


Fig 5(iii)

In case (i), segment PR lies inside the circle, but it is not a complete chord.

In case (ii), the end point R of the extension lies outside the circle. Therefore line QR joins a point inside to a point outside, which means it crosses the circumference – let us say at point T. In that case PT is a chord of the circle (i.e. it satisfies the definition of a chord), and it lies wholly inside the circle.

In case (iii), PR is a chord of the circle, and lies inside the circle.

Thus two of these three cases yield a chord, and in both of these the chord lies inside the circle, demonstrating the Theorem.

PROPOSITIONS

A14 Corollary If a straight line intersects a circle it does so twice.

For if a straight line intersects a circle it must have at least one point inside the circle and one point on the circumference.

And in that case the considerations of the previous proof apply. Hence the straight line includes a chord inside the circle, etc., demonstrating the Theorem.

PART B

ANGLES, PARALLELS, TRIANGLES & QUADRILATERALS

ANGLES

Proposition B1 Theorem An angle is a magnitude.

For part of an angle in no way differs from the whole save in size and position. [Definition 7]

Proposition B2 Theorem An angle considered clockwise is equal to itself considered anticlockwise.

For they are constructed identically. [A2 Theorem]

Remark A system is chosen here in which angles and other magnitudes are not distinguished by direction.

Proposition B3 Theorem If a straight line divides an angle into two adjacent angles, the latter together make up the original angle.

This follows from Definition 27.

Proposition B4 ***Theorem*** Angles which are not adjacent can be added.

For angles are unchanged on being transferred in space. [Postulate]
And once adjacent, with a common centre, the angles can be added. [Definition 27]

Proposition B5 ***Problem*** To drop a perpendicular to a straight line, through a given point not on that line, and to construct right angles.

First, let us suppose that the given point, P, is not on the given line (Fig 6).

Construction Centre P, place the compasses' pen on some point A in line ℓ not too close to P. Draw a circle, intersecting ℓ again at B. Same radius, centre A, draw a circle. Same radius, centre B, draw a circle intersecting the second circle at P'. By construction, these two circles also intersect at P. Join PP', intersecting line ℓ at M (Fig 7)

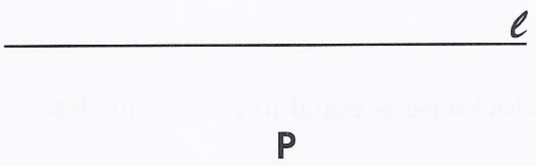


Fig 6

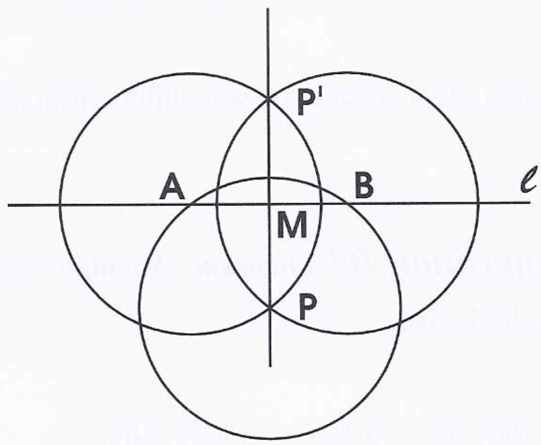


Fig 7

Then PM is perpendicular to line ℓ, and angles PMA and PMB are right angles.

Proof The constructions on either side of line PP' are identical, so that angles AMP and BMP have been constructed identically.
That is angle AMP = angle BMP. [A2 Theorem]
Therefore they are right angles, and PM is perpendicular to line ℓ. [Definition 26].
This demonstrates the Proposition, if the given point is not on the given line.

Secondly, if P lies on ℓ the construction differs only in that the second and third circles need to have a radius r, greater than that of the first circle. Points P and M now coincide, and $P'P$ is perpendicular to ℓ. The proof is essentially the same.

Proposition B6 Theorem A complete turn equals two half turns.

For in completing a turn [Definition 18] two half turns are made [Definition 19], the first in going to the opposite direction [Definition 17] and the second in returning from there to the original direction.

Proposition B7 Theorem All complete turns are equal, and all half turns are equal, and all right angles are equal.

For changing the location of a magnitude does not change it. [Postulate]

Proposition B8 Theorem A half turn equals two right angles.

For right angles are formed by dividing a half turn into two equal angles. [Definition 26]

B8 Corollary A complete turn equals four right angles.

Proposition B9 Theorem Vertically opposite angles are equal.

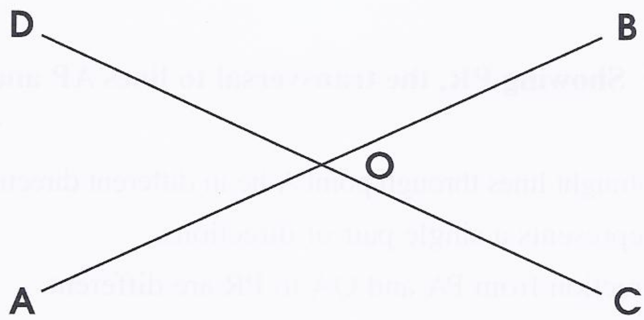


Fig 8

Proof: For let AOB, COD be intersecting straight lines (Fig 8).
Then AO and OB point in the same direction, [Definition 15]
and likewise CO and OD.

Therefore the change in direction from AO to CO is the same as the change in direction from OB to OD.
That is, angle AOC = angle BOD, [Definition 23]
which establishes the Theorem.

Second Proof: The two 'vertically opposite' angles are constructed identically, therefore they are equal. [A2 Theorem]

Remark This Theorem is often referred to as 'vertically opposite'.

PARALLELS

Proposition B10 Theorem Two intersecting straight lines make unequal corresponding angles with a transversal.

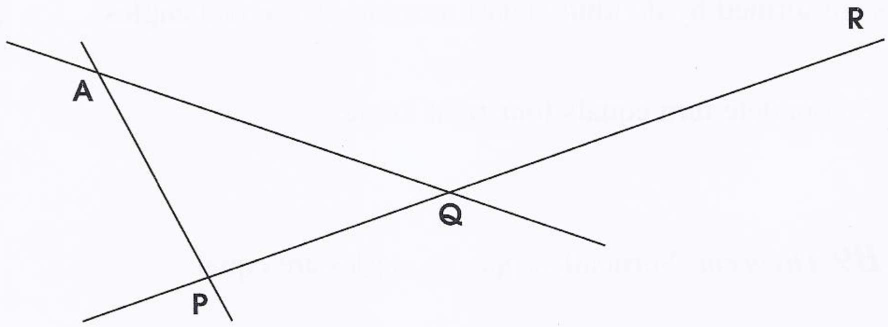


Fig 9 Showing PR, the transversal to lines AP and AQ

Proof: In Fig 9 the two straight lines through point A lie in different directions. [Definition 15]
But the transversal PR represents a single pair of directions.
Hence the changes in direction from PA and QA to PR are different.
Therefore angles APR and AQR are constructed differently.
Therefore they are unequal. [A3 Theorem]
This establishes the Proposition.

PROPOSITIONS

Proposition B11 ***Theorem*** If corresponding angles are equal, the lines intersected by the transversal are parallel.

For if the latter intersect, corresponding angles are unequal. [B10 Theorem]
Therefore since corresponding angles are not unequal the pair of lines do not intersect.
That is, they are parallel, demonstrating the Theorem. [Definition 31]

Proposition B12 ***Theorem*** If a transversal makes unequal corresponding angles with a pair of straight lines the latter intersect.

Proof : For B10 Theorem applies to any pair of intersecting straight lines whatsoever, and between them these encompass all pairs of different directions.
And it applies to any transversal whatsoever.
And angles being bounded or specified by meeting straight lines, [Definition 24]
Proposition B10 applies to all possible pairs of unequal angles.

That is, unequal angles with a transversal only occur if the pair of straight lines intersect.
Being a restatement of the Theorem, this establishes it.
Remark Summarising this proof: inspection shows that all cases of unequal corresponding angles with a transversal arise from intersecting lines.

Proposition B13 ***Theorem*** Parallel lines make equal corresponding angles with a transversal.

For if corresponding angles are unequal, the pair of lines cut by the transversal intersect.
[B12 Theorem]
Therefore parallel lines cannot make unequal corresponding angles with a transversal.
That is, parallel lines make equal corresponding angles with a transversal.

Proposition B14 ***Theorem*** Parallel lines make equal alternate angles with a transversal.

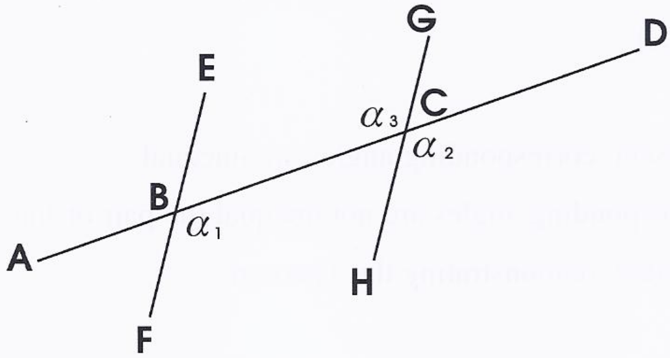


Fig 10 Illustrating alternate angles a_1 and a_3, EF and GH being parallels

It will be shown that $\alpha_1 = \alpha_3$ (Fig 10).

Proof: $\alpha_1 = \alpha_2$ [B13 Theorem]

$\alpha_2 = \alpha_3$ [B9 Theorem]

Therefore $\alpha_1 = \alpha_3$, [A4 Theorem]

and this establishes the Theorem.

Proposition B15 ***Theorem*** If alternate angles with a transversal are equal, the lines are parallel.

For if alternate angles are equal corresponding angles are equal.

[B9 Theorem, 'vertically opposite']

And if corresponding angles are equal, the lines are parallel. [B11 Theorem]

That is, if alternate angles are equal the lines are parallel.

Second Proof In Fig 11, the figures above and below line ℓ are constructed identically.
Therefore they are congruent.
Therefore if j and k intersect above line ℓ they also intersect below it.
But they can only intersect once if they do so at all.
Therefore they do not intersect, i.e. they are parallel.

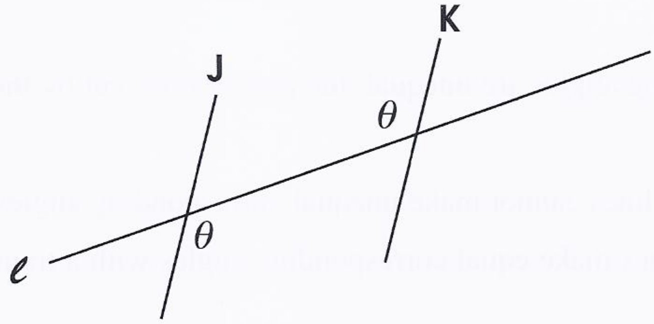


Fig 11

PROPOSITIONS

TRIANGLES

Remark A triangle is specified by three points, since these determine the three sides which bound it. If we are given the base of the triangle, this specifies two points. The third point can be located by two intersecting lines, and the arcs of two circles will do the job nicely.

***Proposition B16* Problem** To construct a triangle with sides equal to those of a given triangle.

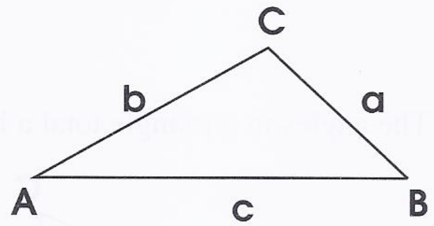


Fig 12 The given triangle

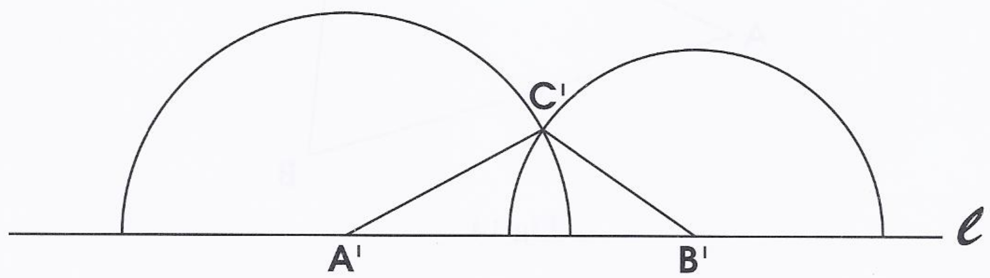


Fig 13 Showing two intersecting semicircles above line ℓ

Given Let a, b and c be the sides of the given triangle ABC, with AB = c, BC = a, CA = b (Fig 12).

Construction Draw a straight line ℓ, and using the compasses, mark a length $A'B' = c$ on it.
Centre A', draw a semicircle of radius b above line ℓ.
Centre B', radius a, draw another semicircle above line ℓ intersecting the first semicircle at C' (Fig 13).
It will be shown that $\Delta A'B'C'$ has the same sides as ΔABC.

Proof $A'C' = b$, since C' lies on the first semicircle, of radius b.
Also $B'C' = a$, since C' lies on the second semicircle.
And $A'B' = c$, by construction.
Therefore, Δ's $A'B'C'$ and ABC have the same sides, as required.

Remark If the two semicircles met on the line instead of above it, they would just touch one another.

B16 Corollary Two triangles constructed with the same three sides are congruent.
For since the two triangles can be constructed identically, they are congruent. [A1 Theorem].

Proposition B17 Theorem The angles in a triangle total a half turn.

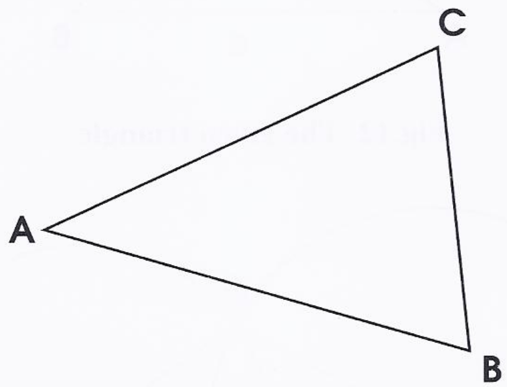


Fig 14

Proof: Consider ΔABC (Fig 14).
Noting that angles which are not adjacent may be added, [Prop. B4]
let the straight line through AB rotate clockwise:
first about B to coincide with CB, then about C to coincide with CA, then about A to coincide with BA.

It performs a half turn in the process, changing direction from AB to BA, having passed successively through all the angles inside the triangle.
Thus the angles of a triangle total a half turn.

Remark It can be objected that, strictly speaking, straight lines do not move.

Proposition B18 Theorem An exterior angle of a triangle equals the sum of the interior opposite angles.

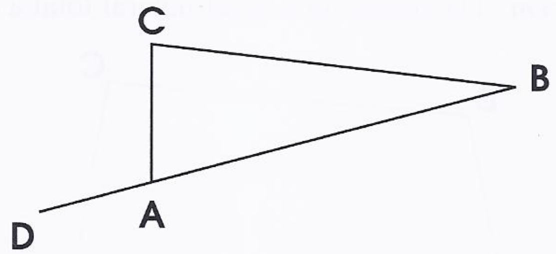


Fig 15 Showing angle DAC, an exterior angle of triangle ABC

Proof: Let the straight line through DAB rotate clockwise, first about B to coincide with CB, then about C to coincide with CA. Thus the rotation has passed through, or summed, the interior angles at B and C. [Prop. B4]

But the change in direction from DA to CA is also measured by a rotation clockwise through angle DAC.
That is, angle DAC = angle ACB + angle ABC, as required.

Proposition B19 Theorem The base angles of an isosceles triangle are equal.

For the two angles are constructed identically. [A2 Theorem]

Remark Isosceles triangles arise, e.g., when two radii of a circle form two of the sides.

Proposition B20 Theorem An exterior angle at the apex of an isosceles triangle is twice an interior opposite angle.

For that exterior angle is the sum of the interior opposite angles, [B18 Theorem]
and the latter being equal, [B19 Theorem]
the exterior angle is twice the interior opposite angle.

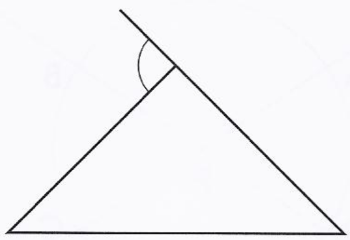


Fig 16 Showing an isosceles triangle and an external angle at its apex.

QUADRILATERALS

Proposition B21 ***Theorem*** The angles in a quadrilateral total a complete turn.

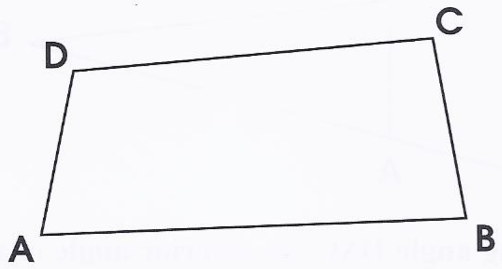


Fig 17 Quadrilateral ABCD

Proof: In Fig 17, let a straight line initially coincident with AB be rotated clockwise,
first about B until it coincides with CB,
then about C until it coincides with CD,
then about D until it coincides with AD,
then about A until it coincides with AB.

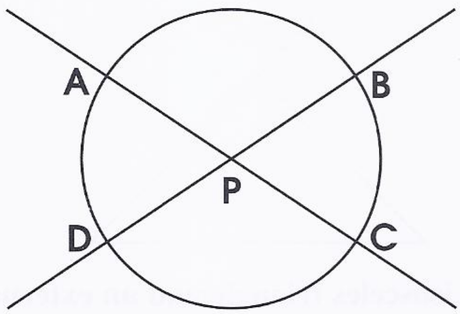
Thus it has returned to the position and direction in which it started, having made a complete turn, and having passed through all angles inside the quadrilateral to do so.

That is, the angles in quadrilateral ABCD together make a complete turn.
[Note that the addition of angles which are not adjacent is demonstrated in Prop. B4.]

Proposition B22 ***Problem*** To construct a rectangle.

Construction Draw two straight lines at P.
Centre P, draw a circle, intersecting the first straight line at points A, B, C & D (Fig 18).
Join AB, BC, CD, DA.
Then ABCD is a rectangle.

Fig 18



Proof All four angles having been constructed identically, they are equal. [A2 Theorem]
Therefore ABCD is a rectangle. [Definition 56]

Remarks (1) It suffices to join AB, BC etc. mentally.
(2) The aim, here, is not to construct a rectangle of particular dimensions; it is to use a simple construction to demonstrate certain results, the first two of which now follow.

B22 Corollary (1) A diagonal divides a rectangle into two congruent triangles.
For the two triangles are constructed identically. [A1 Theorem]

B22 Corollary (2) Opposite sides of a rectangle are equal.
For they are constructed identically. [A2 Theorem]

Proposition B23 Theorem Each angle of a rectangle is a right angle.

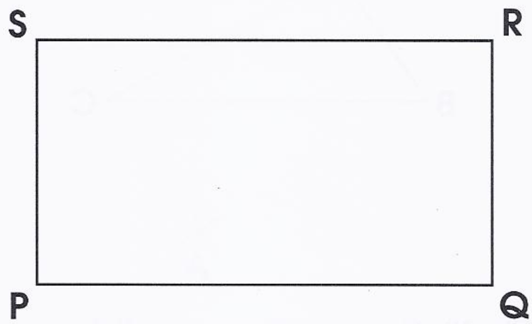


Fig 19 Rectangle PQRS

Proof: The rectangle contains four equal angles. [Definition 56]
And together they make a complete turn. [B21 Theorem]
But a complete turn contains four right angles, [B8 Corollary]
these being four equal angles. [B7 Theorem]
Hence each angle of the rectangle is a right angle.

Proposition B24 Theorem All sides of a square are equal.

For a square is a special case of a rectangle. [Definition 57]
Therefore its opposite sides are equal. [B22, Corollary (2)]
And two of its adjacent sides being equal, [Definition 57]
it follows that all four sides are equal.

PROPOSITIONS

Notation Let a rectangle with sides a and b be referred to as Rectangle (a, b) and a square of side c as Square (c).

Proposition B25 **Problem** To construct a parallelogram, and to construct a parallel to a given line, through a given point.

Construction Draw a triangle, ABC.

Centre B, radius AC, draw an arc.

Centre C, radius AB, draw an arc, intersecting the first arc at A′(Fig 20).

Join BA′ and CA′

Then BA′CA is a parallelogram.

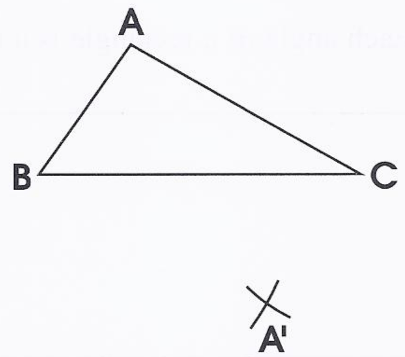


Fig 20 Constructing a parallelogram

Proof Triangles ABC and A′CB are congruent. [B16 Corollary]

Therefore angle ACB = angle A′BC.

Therefore AC is parallel to BA′. [B15 Theorem]

And similarly AB is parallel to CA′, whence BA′CA is a parallelogram.

The same construction applies for the second part of the Proposition, on letting C be the given point, and AB the given straight line.

Remark Note the principle used here: two congruent triangles asymmetrically placed on opposite sides of the same base give rise to a parallelogram.

B25 Corollary (1) A diagonal divides a parallelogram into two congruent triangles.

B25 Corollary (2) Opposite sides of a parallelogram are equal.

Remark How can a *problem* have corollaries? The latter are consequential theorems, which follow once the construction is established.

Proposition B26 Theorem A parallelogram with a right angle is a rectangle.

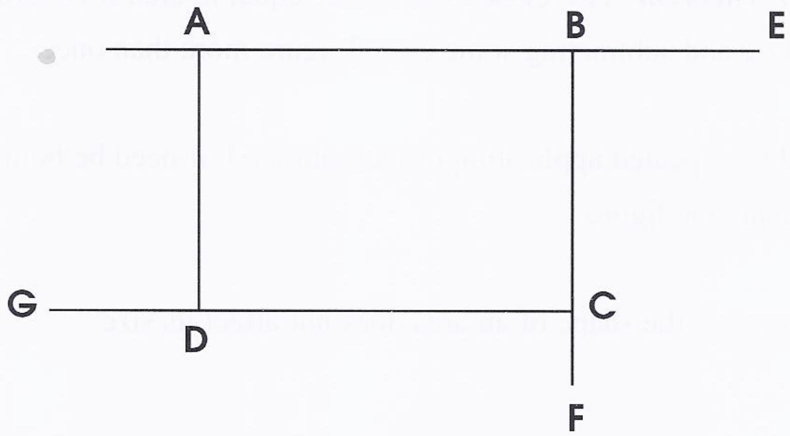


Fig 21 Showing a parallelogram ABCD

Proof: Given that BAD is a right angle,

Angle EBC = a right angle.	[B13 Theorem]
Therefore angle ABC = a right angle.	[B8 Theorem]
Hence angle FCG = a right angle.	[B13 Theorem]
Therefore angle GCB = a right angle.	[B8 Theorem]
Hence angle GDA = a right angle.	[B13 Theorem]
Therefore angle ADC = a right angle.	[B8 Theorem]

That is, all four angles inside the parallelogram are right angles.

Therefore they are equal,	[B25 Corollary (2)]
and ABCD is a rectangle.	[Definition 58]

This demonstrates the Theorem.

PART C

CONCERNING AREA EQUALITIES AND SIMILAR TRIANGLES

Definition 64 A closed figure is said to be equal in area to a second figure if the two can be brought into congruence by modifying one of them, by both adding and subtracting a third closed figure.

Proposition C1 ***Theorem*** Two closed figures are equal in area if congruence is brought about by both adding and subtracting some closed figure more than once.

This is established by repeated application of Definition 64, if need be both adding and subtracting more than one figure.

Remark This shows that the shape of an area does not affect its size.

Proposition C2 ***Theorem*** Area is a magnitude.

For a given area and a part of it both possess the property referred to in Definition 64. And in this respect they differ only in size and position, shape being irrelevant.

Proposition C3 ***Problem*** On a given base to construct a rectangle equal in area to a given rectangle.

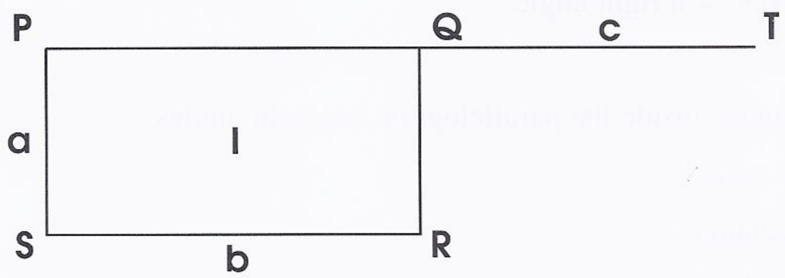


Fig 22 The given rectangle I

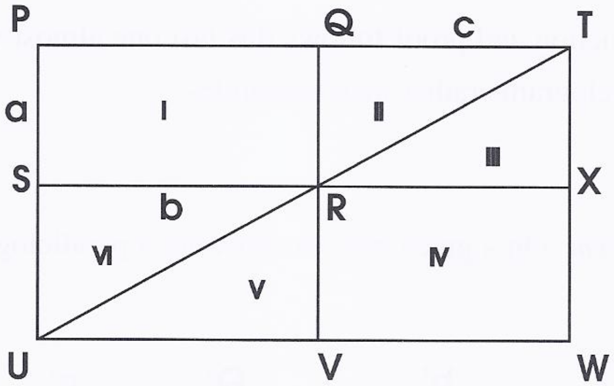


Fig 23 Showing how the rectangle IV is constructed

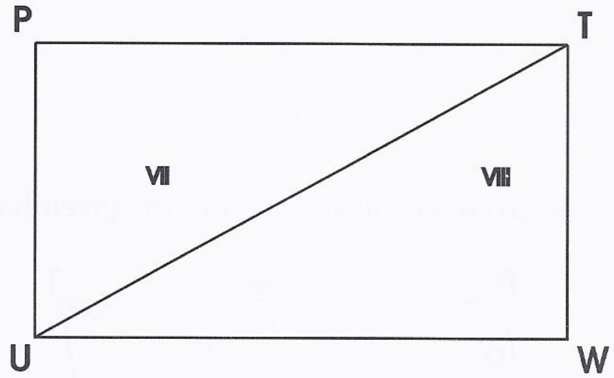


Fig 24 A simplified version of Figure 23

Given PS = a and SR = b are the sides of the given rectangle, PQRS, and c is the given base on which a rectangle of equal area is to be constructed, where QT = c and PQT is a straight line (Fig 22).

Construction Extend PS to meet TR extended at U [Provision 2]
Draw a straight line through T, parallel to PU, and another through U, parallel to PT, to meet at W. [Proposition B25]
Extend QR to meet UW at V.
Extend SR to meet TW at X [Provision 2].
(see Figure 23)

Then IV is the required rectangle, equal in area to Rectangle I, with sides c and d (say).

Proof All the parallelograms so constructed are rectangles, since they each have one right angle. [B26 Theorem]

Also, in Figures 24 and 23: (1) Area VII = Area VIII [B25 Corollary (1)]
(2) Area VI = Area V [B25 Corollary (1)]
(3) Area II = Area III [B25 Corollary (1)]

Adding (2) and (3) and subtracting from (1), Area I = Area IV, as required.

Remark The next construction and proof follows this last one almost word for word, differing in that it applies to parallelograms rather than rectangles.

Proposition C4 **Problem** On a given base to construct a parallelogram equal in area to a given parallelogram.

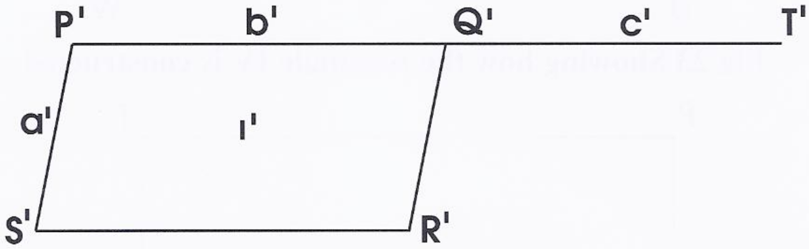


Fig 25 The given parallelogram and the given base C'

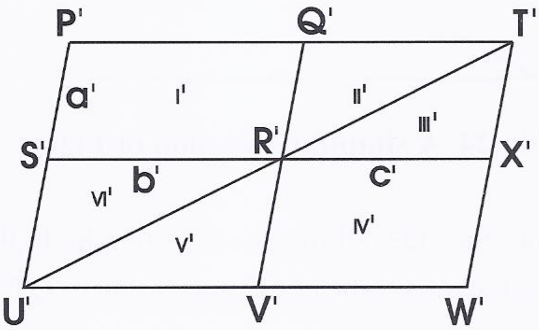


Fig 26

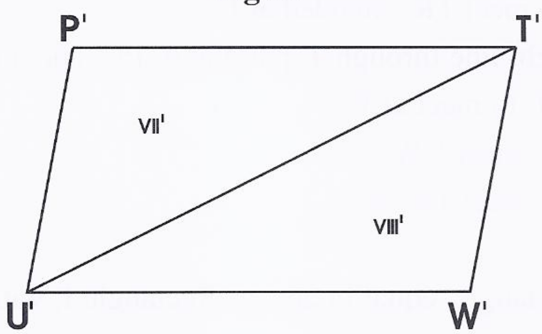


Fig 27

Given Figure 25, Figure 26 is constructed from it in the same manner as in Proposition C3. And using dashed letters this time, the proof likewise follows that of Proposition C3.

Proof (1) Area $\mathrm{VII}' = $ Area VIII' [B25 Corollary 1]

(2) Area $\mathrm{VI}' = $ Area V' [B25 Corollary 1]

(3) Area $\mathrm{II}' = $ Area III' [B25 Corollary 1]

Adding (2) and (3) and subtracting from (1), Area $\mathrm{I}' = $ Area IV', as required.

PROPOSITIONS

Proposition C5 ***Theorem*** If two parallelograms have the same angles and equal areas, the rectangles on their sides are equal in area.

This follows from Propositions C3 and C4, on putting $a = a'$, $b = b'$, $c = c'$, $d = d'$, the first of these two propositions being a special case of the second.

In more detail, the rectangles on sides a, b and c, d are equal in area by Proposition C3, and the parallelograms on the same sides are equal in area by Proposition C4.

Proposition C6 ***Theorem*** Selecting from two pairs of corresponding sides of similar triangles, the rectangles on non-corresponding sides are equal in area.

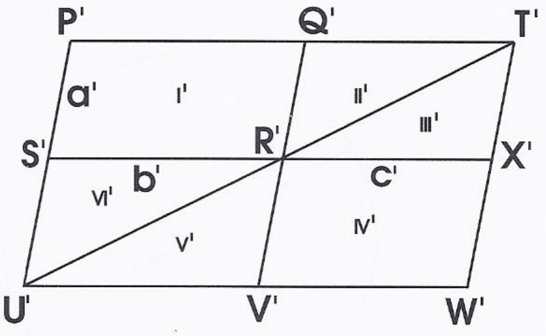


Fig 28a Figure 26 of Prop C4, putting $X'W' = d'$

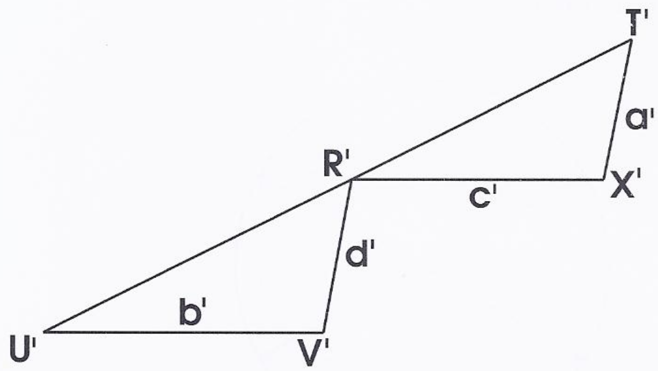


Fig 28b An extract from Fig 26, showing similar triangles $U'V'R'$ and $R'X'T'$

Proof In Fig 26 of Proposition C4,
angle $U'V'R'$ = angle $R'X'T'$,
since both angles are equal to angle $U'W'T'$ by Proposition B13.
And angle $R'U'V'$ = angle $T'R'X'$ by Proposition B13.
Therefore triangles $R'U'V'$ and $T'R'X'$ are similar. [Definition 53]

Also, $U'V' = b'$, $V'R' = d'$,
$X'T' = a$, and $R'X' = c'$ [B25 Corollary (2)]
(See Figure 28).
In the similar triangles of Figure 28, a' and d' are corresponding sides, and so are c' and b'
[Definition 33]

Now by Proposition C5,

Area of Rectangle (a′, b′) = Area of Rectangle (c′,d′).

That is, restricting consideration to two pairs of corresponding sides of the given similar triangles, the rectangles on non-corresponding sides are equal in area.

PART D

ELEMENTARY PROPERTIES OF A CIRCLE

Proposition D1 **Theorem** A diameter of a circle subtends a right angle on the circumference.

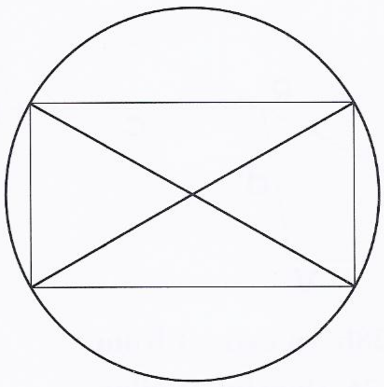


Fig 29

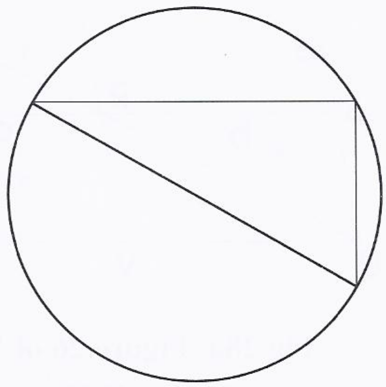


Fig 30

Proof The procedure given in Proposition B22 for constructing a rectangle is indicated in Fig 29. By B23 Theorem, all four angles of a rectangle are right angles.

And on omitting parts of Fig 29, there remains a diameter subtending a right angle on the circumference (Fig 30).

This demonstrates the Theorem.

Proposition D2 **Theorem** Opposite angles of a cyclic-quadrilateral total a half turn.

First note that in Figures 31 and 32 a cyclic-quadrilateral is divided into four isosceles triangles, with base angles α, β, γ and δ. ‘O’ is the centre of the circle.

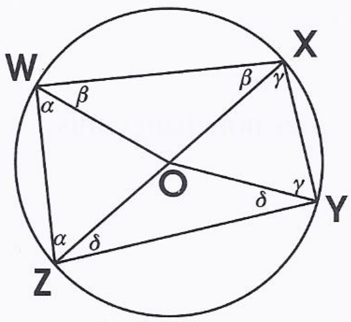


Fig 31 Case I, 'O' inside a cyclic-quadrilateral

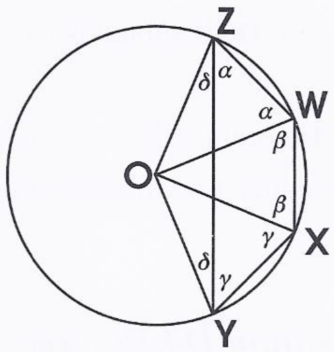


Fig 32 Case II, 'O' outside a cyclic-quadrilateral

Proof By B19 Theorem the angles in a quadrilateral total a complete turn.

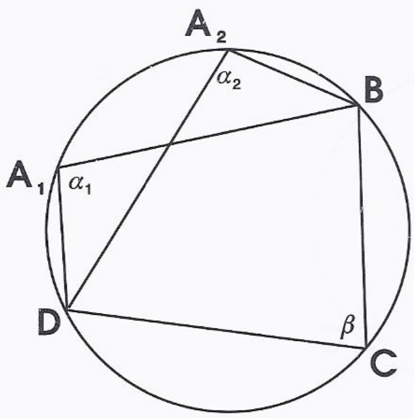
CASE I It can be seen in Figure 31 that angles W +Y and Z+X both come to $\alpha + \beta + \gamma + \delta$.
That is, opposite angle-sums are equal
Therefore each must equal a half turn.

CASE II Opposite angle-sums in Figure 32 total $\alpha + \beta + \gamma - \delta$ in each case.
That is, angle ZWX + angle XYZ = $\alpha + \beta + \gamma - \delta$ = angle WXY + angle YZW

Once again opposite angle-sums are equal to one another, and when added make a complete turn. Therefore each is a half turn.
This establishes the Theorem in both cases.

Proposition D3 ***Theorem*** Equal arcs of a circle subtend equal angles on the circumference.

Fig 33



Proof In Fig 33, $\alpha_1 + \beta = \frac{1}{2}$ turn [D2 Theorem]
$\alpha_2 + \beta = \frac{1}{2}$ turn [D2 Theorem]
Therefore $\alpha_1 = \alpha_2$

That is, arc DCB subtends a constant angle on the circumference.

By A2 Theorem, wherever arc DCB is constructed on the circle does not change this,
That is, equal arcs subtend equal angles on the circumference.

Proposition D4 ***Theorem*** An exterior angle of a cyclic-quadrilateral equals the interior opposite angle.

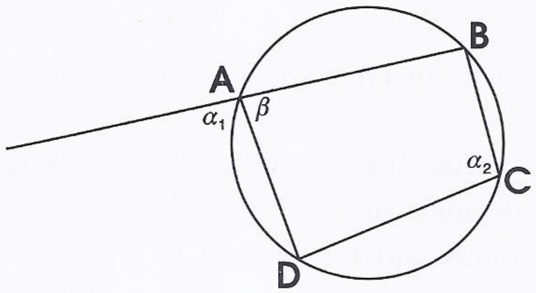


Fig 34

Proof $\alpha_1 + \beta = \frac{1}{2}$ turn [Proposition B3]

And $\alpha_2 + \beta = \frac{1}{2}$ turn [Proposition D2]

Therefore $\alpha_1 = \alpha_2$, [Propositions A4 and A6]

establishing the Theorem.

Proposition D5 ***Theorem*** The angle between the tangent and the chord equals the angle subtended by that chord on the far arc.

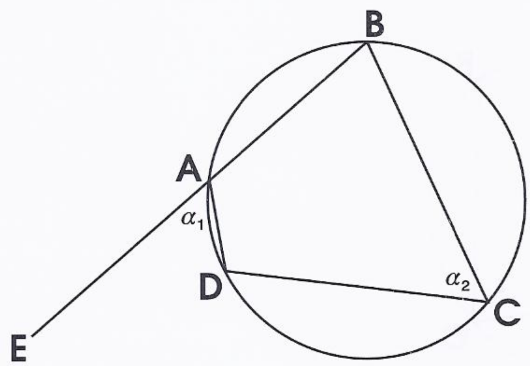


Fig 35

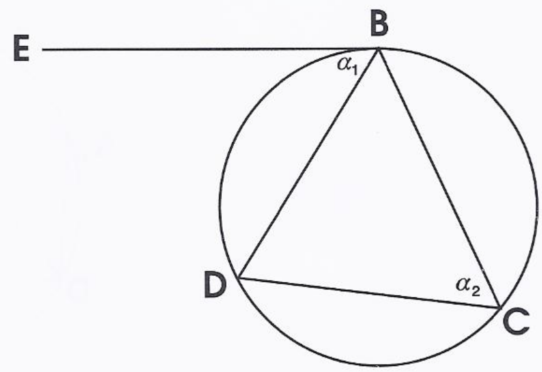


Fig 36

In Fig 35 suppose that A is a point on arc DAB somewhere between B and D, with $\alpha_1 = \alpha_2$ [D4 Theorem]

Let positions of A approach and then coincide with point B (Figure 36). When this happens the straight line EB becomes the tangent at B. [Definition 41]
Since $\alpha_1 = \alpha_2$ throughout, it follows that the angle between the tangent and the chord (DB) equals the angle subtended by that chord on the far arc.

This completes the proof.

Remark Examples of this 'limiting case' type of proof are to be found in *Discover Vedic Mathematics* by K. Williams.

Proposition D6 Theorem The angle at the centre of the circle equals twice the angle subtended at the circumference.

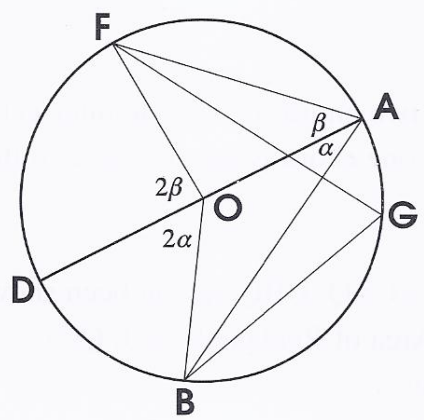


Fig 37 Showing a circle, centre O, diameter DOA

In Fig 37, G is any point on arc FAB.
It will be shown that angle FOB = 2 × angle FGB, so demonstrating the Theorem.

Proof Δ's OAF, OBA are isosceles,
Therefore angle FOD = 2 × angle FAD, [B20 Theorem]
and angle DOB = 2 × angle DAB [B20 Theorem]
By addition, angle FOB = 2 × angle FAB
But angle FAB = angle FGB [D3 Theorem]
Therefore angle FOB = 2 × angle FGB
This demonstrates the Theorem.
Note that only the one case is needed here because $2\alpha + 2\beta$ can take any value between a complete turn and nothing.

Proposition D7 ***Theorem*** The triangles formed from internally intersecting chords of a circle are similar.

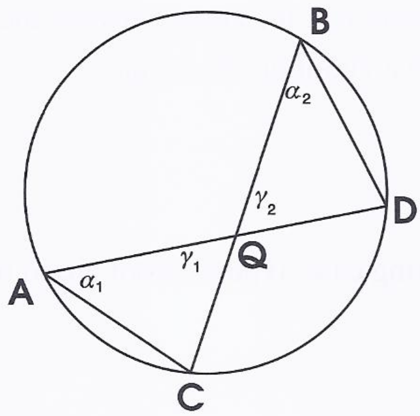


Fig 38

Proof In Fig 38, $\alpha_1 = \alpha_2$ [D3 Theorem]
and $\gamma_1 = \gamma_2$ [B9 Theorem]
Therefore Δ's CAQ, DBQ are similar. [Definition 54]

Proposition D8 ***Theorem*** If two chords of a circle intersect one another internally, the rectangle on the two parts of the one chord is equal in area to the rectangle on the two parts of the other chord.

For in Proposition Δ 7, Fig 1, Δ's CAQ, DBQ having been shown to be similar,
Area of Rectangle (AQ, QD) = Area of Rectangle (BQ, QC), [C6 Theorem]
which demonstrates the Theorem.

Proposition D9 ***Theorem*** Where a chord is intersected externally by a tangent, the rectangle on the two parts of the chord is equal in area to the square on that tangent.

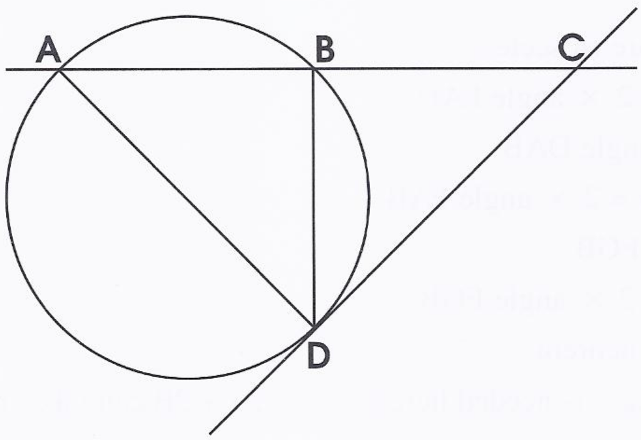


Fig 39 The tangent at D meets chord AB externally at C

Proof In Fig 39, angle CDB = angle DAC [D5 Theorem]

Therefore having C in common, Δ's ADC, DBC are similar. [Definition 54]

Hence Area of Rectangle (AC, BC) = Area of Square (DC) [C6 Theorem]

This establishes the Theorem.

Proposition D10 Theorem Where two chords of a circle intersect externally, the rectangle on the two parts of the one chord is equal in area to the rectangle on the two parts of the other chord.

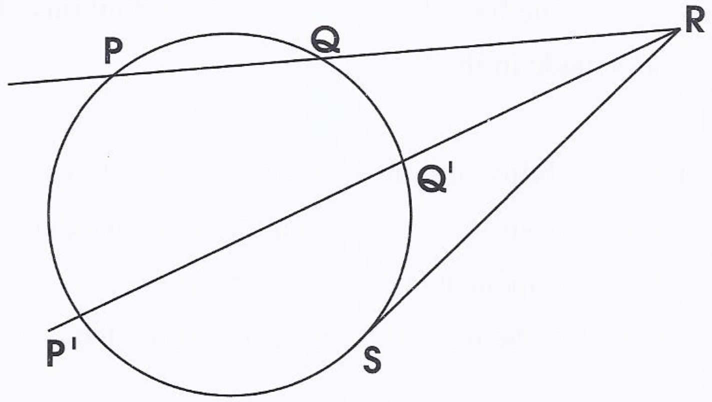


Fig 40

Proof For C6 Proposition applies equally to chord PQR, tangent SR, and to chord $P'Q'R$, tangent SR (Fig 40).

That is, Rectangle (PR, QR) and Square (SR) are equal in area, and so are Rectangle ($P'R$, $Q'R$) and Square (SR).

Therefore Rectangle (PR, QR) and Rectangle ($P'R$, $Q'R$) are equal in area.

This establishes the Theorem.

CONCLUSIONS

What has been achieved in this book?

1. Proofs of elementary properties of a circle have been given which can swiftly be followed mentally. Such a proof is like a key, given which the theorem follows. For example, recognizing that a diagonal of a rectangle is a diameter of the circumscribing circle is a key. The immediate conclusion is that a diameter subtends a right angle on the circumference of the circle.

2. A system has been set up placing the theorems in context, giving them a foundation. And this latter essentially consists of the four Provisions and the Definitions, these being used in accordance with the remarks made in the Preliminaries.

3. Thus in this system, the availability of drawing instruments and material is granted, their roles being spelled out by definitions. Provision 2, on the one hand, and Provisions 1, 3 and 4 on the other, are like different components in the study of geometry. From the former there spring figures, and from the latter the means of formulating and demonstrating their properties.

4. Movement is incorporated into the system via the Postulate. This leads easily to the necessary theorems on parallels.

COMMENTARY

INTRODUCTION

This Commentary was written to anticipate queries and to act as a tool for evaluating the approach of this book. Above all, it seeks to demonstrate that by meeting the needs of an oral tradition, a coherent approach to the study of geometry can be developed.

Also, there are all sorts of questions and points which can arise in connection with a piece of mathematics such as this one, and the Commentary gives a place in which to air them. The following short essay gives a taste of this, as well as sketching the background to some of the topics covered in the Commentary. It begins with an apparently straightforward question.

Why not use Euclid's definitions, instead of supplying others?

Euclid's definitions served the *Elements* well, and are part and parcel of an impressive achievement. When written the *Elements* was a thoroughly modern work, produced under the influence of the prevailing climate of thought. There being no dictionaries in those days, an important function of the definitions was to ensure clarity as to the meaning of key words used in the study.

Today there is a different intellectual climate. Ideas concerning Euclidean geometry went through a revolution in the 19th century, in the wake of the emergence of non-Euclidean geometries. Formerly, most mathematicians would probably have considered it obvious that there is only one straight line through a given point and parallel to a given straight line. Now it became accepted that there might be none, or more than one, depending on the type of geometry.

The question was, what else that had been widely considered to be self-evident might prove not to be in fact so? A new awareness arose of assumptions, and the need to question them. What right had one to assume anything, other than what was given as an axiom or postulate?* Did that mean that more were needed?

---

* In the resulting maelstrom of ideas, somewhere along the line a number of mathematicians ceased to use the word *axiom* in its traditional sense, as a self-evident statement that is granted. For some reason the term *postulate* did not come to the fore in its place – perhaps stemming from uncertainty as to what Euclid's corresponding term meant.

As for definitions, what was being assumed in them? An ideal emerged that **all** terms of a geometric nature be defined, not just a few that were judged to be key ones. For example, Euclid defines a line as having length but not breadth. But what is *length*, and what is *breadth*? Euclid's set of definitions was not designed to cope with such requirements, and therefore could not be part of the new thinking. A fresh look was required.

Influenced by these and other considerations, there was a tendency for geometry to become more abstract in the late nineteenth century. The movement culminated in David Hilbert's *Gründlagen der Geometrie*, published in 1899 [*Foundations of Geometry*]. Amongst other things it was intended to tackle problems arising connected with the aforementioned changes in thinking concerning geometry. But its especial merit was in casting light on questions of great interest to mathematicians: how do we know that the axioms of geometry are consistent with one another, and that they are independent of one another?

This was its strength. Its weakness lay in the difficulty in using it to prove ordinary theorems of geometry – not least because it dispensed with diagrams (in principle).*

Since then the belief has grown and spread amongst mathematicians that in order to tackle geometry properly it needs to be done without diagrams. This stems from Hilbert's above-mentioned work, which when it came out aroused vigorous opposition, but later won the field because it was able to tackle such important questions, and because it was considered to be rigorous – at least initially. There have been a number of other formulations on similar lines this century, presumably in pursuit of rigour.

Geometry for an Oral Tradition is not simply presenting a way of thinking which could have existed before writing. **There is a definite attempt to set it in the modern context** – to update it – so enabling mathematicians to take it seriously, as a valid approach today.

---

* And that almost certainly arose in this way. Attempting to define all terms of a geometric nature proved not to be easy; for each of them is usually defined with the aid of one or more others. But that is the sort of thing which can go on and on – or else round in a circle. Where are we to start? Very likely it was in order to cut across this that Hilbert proposed starting with three undefined terms: *point, line* and *plane*. That is, they are not assigned their ordinary meanings. In that case, what sense can be made of figures? Hilbert decided to dispense with them.

There is a further point in connection with the opening question, however. As well as general considerations concerning current ways of thinking, there are the special requirements of this particular study. These preclude the use of at least two of Euclid's definitions (of *angle* and *figure*), and possibly more. [See *The Rationale of eight key definitions*, Commentary, Part II, 3(g).]

To sum up, there are two reasons for not using Euclid's definitions: firstly, the needs of this particular study are different from Euclid's, and secondly there is a very different climate of thought today from that in Euclid's time. To set this material in today's context, a different set of definitions is required.

OVERVIEW

The following are amongst the more important points in the Commentary:

(i) The significance of the distinction between stated and unstated assumptions is discussed. [Part I]

(ii) In this system language is shown to play an especially important role. [Part II]

(iii) It is shown how Provisions 1 and 3 are related. [Part II]

(iv) The background is explained, to the selection of a criterion for selecting words to be included in the list of definitions. [Part II]

(v) Some important objections to the inclusion of movement in geometry are noted and answered. [Part IV]

(vi) Since figures are central to a study of geometry such as this one, there is a need to ensure that they are used validly. What are the relevant rules? Some are known, others are proposed;

more accurately, they may all be known, but do not appear to have been assembled in one place, and a step towards this is attempted here. [Part V]

(vii) The nearest equivalents in the present text to Euclid's five common notions and five postulates are given. [Part III]

PART I SOME BASICS

1. STATED AND UNSTATED ASSUMPTIONS

If assumptions or agreements are stated they become part of the formulation: it gives them status. Unstated agreements, by contrast, creep in by the back door. They may be more or less acceptable but they have not been stated as part of the set-up.

And that is the reason for stating the Provisions. If they were not mentioned they or suitable alternatives would need to be assumed just the same. Being unstated assumptions they would be points of weakness in the chain of reasoning – running the risk of allowing objectionable features into the system.

But, one might object, does it matter if Provisions 1 and 3 are left unstated? Possibly it does not. The question is, is it or is it not acceptable to have unstated assumptions in a piece of reasoning? An unstated assumption is a potential source of weakness. If the aim is rigour, unstated assumptions are not acceptable.

But if it is argued that some unstated assumptions are harmless* and others are not, what criterion is there for distinguishing between them?

---

* In this context, *harmless* = not at risk of undermining the soundness of the argument.

Is it possible to have such a criterion? Usually assumptions need to be recognised and stated individually. There does not seem to be a formula available for flushing them out. So if there is no known pattern to them, what hope is there of finding a criterion for selecting the harmless ones?

If it is argued that a judgement can be made, as to whether a given assumption is harmless or not, then agreed, so it can. Of course, the judgement may be mistaken.

The same point (about making a judgement) could at one time have been made about Euclid's method of superposition. It seems it was a method in common use (Euclid calls it a *common notion*), and generally considered to be harmless enough. True, it did not abide by the Pythagorean ideal of excluding movement from geometry, but a blind eye was turned to that. The mathematicians could get away with it, though the philosophers could not. Yet more than two thousand years later it was shown that there are circumstances in which it is not valid.

Not only does honesty require us to state any assumptions we are aware of, respect for certainty in the reasoning requires it.

* *

From the students' point of view, unstated assumptions have a serious drawback. They are left to guess what is being assumed – and they may or may not be aware that there is an assumption. This does not make for clarity. There is something they are not being told, and at some point they may well sense this. Furthermore, it might not be straightforward. If it is, of course, then no problem. But how are the students to know? The sense of something lurking and unmentioned is uncomfortable – it might even raise fears of the unknown for some, lack of clearly stated reasoning bringing out the irrational. Others might wonder what it is they are not being told. Might it be something so difficult that it is better not to mention it?

To conclude, it is much better not to have lurking, unstated assumptions. This is more satisfactory because it is more rigorous, because it is honest, and because it keeps the elements of the investigation clear and open, making things easier for the student.

2. THE MATERIAL AND INSTRUMENTS OF GEOMETRY

For example, a sheet of paper is commonly used as a material of geometry, and a pen, straight edge and compasses are amongst its instruments. But more instruments are to be found in the Provisions, and so let us start there.

Reason and language are brought to bear on the subject by mankind (Provisions 3 and 1). Provision 4, the Postulate, can be thought of as coming from geometry, and Provision 2 from the interaction between them. But as the following diagram representing these points shows, the study of geometry uses more instruments than this.

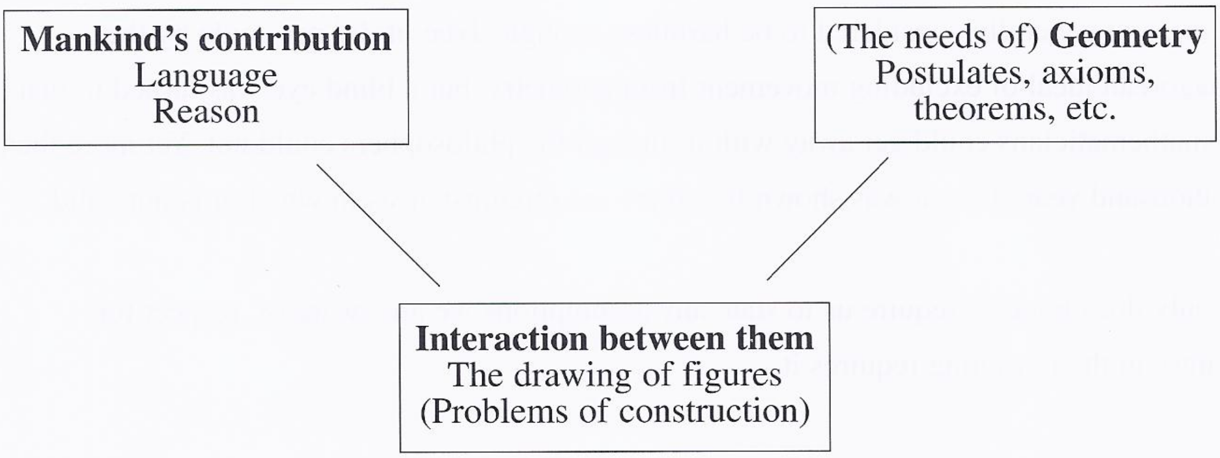


Diagram 1 Allocation of the material and instruments of geometry to their chief sources.

It is quite instructive to take one or two simple examples of proofs, and to observe the need arising for one or other of the instruments of geometry. The following example gives the idea.

Suppose a group of people, untutored in geometry, encounter a path tiled in squares, and are fascinated by its simplicity and regular order. They could, if so inclined, observe that all the angles are equal, and may be led by observing the patterns to the considerations shown in Diagram 2.

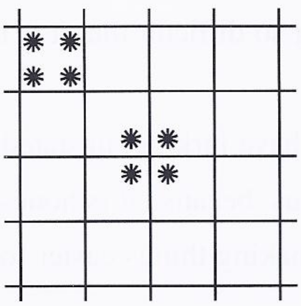


Diagram 2 Patterns of angles noted, within a simple pattern of squares.

There are four of these angles in a square, and also four of them clustered around a point. Being equal angles, it follows that the angles in a square, taken together, cluster neatly around a point. As we might put it, the angles in a square total a complete turn.

There is an insight here, giving rise to a recognition of something new – a piece of knowledge. The latter we call a *theorem**. It is a statement of something we did not start off with, but arrived at. When dealing with more than one example such terms begin to be needed, so that the study can be formulated systematically. A *theorem* is one of the instruments of geometry.

Having begun in this way it soon becomes apparent that other terms (instruments) are needed. E.g. there is a need to state what a square is, and such a statement is called a *definition*.

Also, how was it recognised that all the angles in the pattern of squares are equal? Was it the recognition that they are constructed in the same way? Did our group of people get a chance to play with the tiles, and to recognise that each one fits equally well if turned round, or if turned over, and that any tile can fit in any one of the squares? The reader may have picked up hints of Euclid's method of superposition or of Provision 4 in this second approach, and of Proposition A2 in the first: magnitudes which are constructed identically are equal.

* *

To turn to another matter, for those interested in following up the distinction between materials and instruments the following points may be noted.

1. A material is something to be worked on and an instrument (or tool) something to be worked with. The one has a passive role and the other an active one.

2. **Examples**

Context	**Material**	**Tools or instruments**
Woodwork	Wood	Saw, chisel
Geometry	Paper (plane)	Pen or pencil, compasses

---

* Note the similarity to the Greek word *theos* = god. Doubtless this is because the Greeks thought of knowledge as being of divine origin.

COMMENTARY: PART II

3. However, whatever is in the process of being fashioned is the material currently being worked on. If a tool is being made, it is playing the passive or material role until such time as it is ready for action. Likewise if a theorem is in course of being proved, it is currently the material being worked on, the *matter* in hand. Thereafter it is an instrument.

4. Language is both an instrument for the expression of thought and a storehouse of concepts – a source of material. [A dictionary presents the material aspect of language, using language as an instrument for conveying the thoughts needed.]

PART II LANGUAGE AND REASON

" . . . educated people are apt to forget that language is primarily speech."

Otto Jespersen, *Essentials of English Grammar*

1. GEOMETRY AND LANGUAGE IN AN ORAL TRADITION

How the study begins

Representing some of the simplest forms observed around us in figures, geometry is concerned with studying these, language being its principle instrument.

STARTING POINT

FIGURES → **DEFINITIONS related to figures** ← **LANGUAGE** → **other definitions**

Children learn a lot about geometry at an early age, through play.* Later, the more formal study is begun having learnt a language, and being familiar with figures – perhaps having

* I am told they are fascinated by the five Platonic solids.

learnt to draw them with straight edge and compasses. And we start knowing all the words used to describe simple figures. Indeed we start knowing much more, for all dictionary definitions are available, and apply until they are replaced.*

Now for the next step in the study. For various reasons it is convenient to replace or restate dictionary definitions for some key words – sometimes in order to use the new definition in proofs. Hence the next step: a list of definitions (or perhaps we should call them 'redefinitions').

The two stages of study

The study continues, but a pattern is beginning to be evident which is worth noting. For it emerges that a study such as geometry takes place in two stages:

> first a language is learnt;
> secondly, using relevant words contained in the language, the latter is used to formulate a study of geometry.

It is possible that this is the origin of the division of education into primary and secondary stages.

How a study enriches language

How does it come about that a language contains within it the words needed to study geometry? In the first place, the language is enriched by the study of geometry. This puts the language in the position that the study of geometry arises from it. A diagram illustrates this process:

---

* What is the equivalent of a dictionary in an oral tradition? Whatever it is, calling it a *dictionary* is convenient for these purposes.

COMMENTARY: PART II

LANGUAGE **THE STUDY OF GEOMETRY**

ENRICHED LANGUAGE wider vocabulary, incorporating words such as 'axiom' and 'theorem'

A FORMULATION OF THE STUDY OF GEOMETRY

That is, *the language becomes a repository* for concepts, building upon which an exposition of geometry can be formulated.

2. REASON AND LANGUAGE

2(a) THE LANGUAGE OF PROOFS

A proof can be spoken: it uses language, sometimes differing little from everyday speech. For example, here is a line of proof:

> Length AB is equal to length CD, they being opposite sides of a rectangle.
>
> This line (or step of the proof) is also a sentence, although it can be broken into two sentences thus:
>
> Length AB is equal to length CD.
>
> This is because they are opposite sides of a rectangle.

The reason or justification of a step in a proof is often expressed in another theorem. For example, suppose we have established

> Theorem X: opposite sides of a rectangle are equal.

The above line of the proof can then be put,

> Length AB is equal to length CD [Theorem X].

COMMENTARY: PART II

A few steps of proof may combine to establish a theorem.

The analogy in language is a *paragraph*, a few sentences being used to make a point.
A series of theorems (and therefore of proofs) may be used to establish a result (a theorem).
If so, they resemble the use of a series of paragraphs to establish some larger point, as in an essay or reasoned argument.

Short-hand

Mathematical symbols provide short-hand. For example,

AB = CD [Theorem X].

This is clearly a simpler way of writing the line of proof discussed earlier. Its great advantage is that it states the essentials.

Of course, in an oral tradition such a line would be spoken. That being so, what need does an oral tradition have for such short-hand?

Whatever the answer to this question, certainly it is useful in the present book!

Longer proofs

A few steps in a longer proof may combine to form a sub-theorem, or lemma, a result of too little generality to be considered as a theorem in its own right.

Counting this sub-theorem as a paragraph, the whole proof may be likened to an essay, presenting a reasoned argument.

Note that the form described here is essentially the same as that for a sequence of proofs (or theorems).

In law courts and in lectures, lengthy reasoned argument is not unknown (lectures commonly being accompanied by visual display of information). Even today, the oral tradition is not totally absent.

COMMENTARY: PART II

Naming key theorems

A device used in teaching geometry to children is to give a short-hand reference to key theorems – names such as 'vertically opposite', 'corresponding angles', 'two sides and the included angle', etc. This makes it easy to refer to them; also, it tells us that they are important theorems.

This device is undoubtedly useful in an oral tradition.

2 (b) TO WHAT EXTENT ARE PROVISIONS 1 AND 3 INDEPENDENT OF ONE ANOTHER?

(i) What are the similarities and differences between reason and language?

How reason and language are related is a matter of some interest, and is examined next.

First consider THE NATURE OF LANGUAGE. Here are three things that characterize it.

First characteristic. One description of language is that it consists of *speech, listening* and *understanding*, this last taking place mentally. The words take one to the brink of understanding, but the understanding itself lies beyond them.*

Second characteristic. It uses sentences, each of which is intended to convey a single thought.**

Third characteristic. The primary purpose of a language can be explained in terms of the purpose of a sentence: it is to enable two people to share a common thought.

Next, WHAT IS THE NATURE OF REASONING? This, too, is now characterized by three things.

First characteristic. Reasoning can be described as consisting of *speech, listening* and *making inferences*. What is inferred is called the *conclusion*.

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* This makes it possible to rephrase what is said.

** The Shorter Oxford Dictionary defines a sentence as *a series of words in connected speech or writing, forming the grammatically complete expression of a single thought.*

COMMENTARY: PART II

Second characteristic. The argument and the conclusion can be spoken, *but recognizing that the conclusion is valid is done mentally*. It is a contribution made by the speaker in drawing that conclusion, and made independently by the listener in recognizing that the conclusion is valid. This, of course, is the role of Provision 3.

Furthermore, note that the listener needs to *understand* the argument in order to recognize that it is valid.

Hence, reasoning shares in common with language the elements of *speaking, listening* and *understanding*.

Third characteristic. If two statements imply something, then *reasoning is the act of recognizing what they jointly imply* (the *conclusion*).

An example is, firstly recognizing that turning a pencil sufficiently to point in the opposite direction is making a half turn, and then recognizing that on rotating the pencil through the three angles of the triangle in succession, it points in the opposite direction. The conclusion is that the angles of a triangle total a half turn.

What does ordinary language possess that is analogous to this? Well, for a start, in speech it is normal to use a number of sentences. The comparison with reasoning therefore leads us to ask:

> given that one or more sentences have already
> been spoken and understood, *what does a*
> *further sentence contribute to the meaning*?

This is investigated in the next section. The enquiry has so far shown that reasoning and language share the important characteristics of speech, listening and understanding. Evidently their structure is similar, and yet they appear to differ substantially in respect of the third characteristic of reason. As will now be shown, this apparent difference dissolves away on closer investigation, but not until it has led us to think about language in a way that proves to be highly relevant. And in dealing with this point, it becomes clear where the ability to recognize valid reasoning fits into the picture.

COMMENTARY: PART II

(ii) What does each additional sentence contribute to the meaning?

Example 1 'London. Fog everywhere.' Charles Dickens, *Bleak House.*
Here, the second sentence ADDS something to the picture being built up.* And the succeeding sentences each add more, in turn. This is one type of contribution: each sentence ADDS to the meaning.

In a longer example, i.e. one with more sentences, we might prefer to speak of each additional sentence as *modifying* the meaning. Thus ADDITION and MODIFICATION can be two ways of describing the same thing. But *modification* can also refer to *a replacement of something said*, possibly an updating, as in the next part of a story.

ADDITION and MODIFICATION account for a lot of cases, and that is sufficient for present purposes – there is no need to cover all possibilities.** But a third type is of especial interest to us. It can be called IMPLICATION.

The contribution made by an extra sentence can also be to *imply* something. There might be an act of reasoning involved, or there might be such things as topical allusions, association of ideas, hinting at things, etc. Looked at in this way *implication* includes reasoning but casts a wider net.

First example He is a man, and men are mortal.
The conclusion is implied, and need not be stated. But by not stating it more is hinted at, depending on the context. Saying less achieves more, reason being filled out by the imagination.

Second example 'It is a truth universally acknowledged that a single man possessed of a good fortune must be in want of a wife'. Jane Austen, *Pride and Prejudice*

Like the previous example, this is a compound sentence, and it can be broken down into simpler sentences. A dialogue being a good way of doing this, let two additional characters retell it to us – Belinda and Amelia.

---

* It is acceptable to omit words from a sentence, where the meaning is readily clear without them. In this case the principle has been taken to extremes – and yet the meaning is clear enough. In fact, shortening the sentences heightens the dramatic effect. A fuller version of these highly abbreviated sentences might be, 'The place is London. There is fog everywhere'. Yet do the extra words contribute anything? More is achieved by the shortened form.

** Other contributions made by an additional sentence include *reinforcement of what has been said* (such as affirmation or emphasis) and *attempts to present an alternative version* (e.g. contradiction or correction).

COMMENTARY: PART II

Amelia They say he is single. What is more, he is possessed of a good fortune.

Belinda Then he must be in want of a wife!

Amelia Indeed he must. 'Tis a truth universally acknowledged.

This is certainly an example of implication. There is more going on than is actually said. Belinda's statement is put as though it were a deduction, which it certainly is not. But Amelia, having accepted Belinda's 'conclusion', proceeds to compensate for the dubious logic by elevating it to the status of a universally acknowledged truth. Nothing is being done by half measures here!

I suggest that, so long as we follow what is being said, we register that this is bogus logic. If so, *how can we tell that it is bogus?** Is it because we have the ability to recognize valid reasoning? Briefly put, the answer to this is surely 'Yes'.

These examples point to the conclusion, that *the ability to recognize valid reasoning is an aspect of understanding language*. In other words *Provision 3 is an aspect of Provision 1.* However, acknowledging this in principle does not oblige us to merge them. We can choose to keep them as separate statements. If reason belongs to language, still, we may wish to acknowledge when it is reasoning that is relevant rather than simply communicating.

One further point concerning understanding and reason: we have no reason to suppose that *the nature of understanding* varies according to the language used, although we do have more difficulty in an unfamiliar language. The context changes, but not the faculty. What changes is *how* it operates.

And having concluded that the ability to recognize valid reasoning is an aspect of understanding, the same point applies to reasoning: we have no reason to suppose that the faculty itself varies according to the context, although in some it operates easily and in others less so. [Of course, to develop reasoning ability, plenty of practice in simple contexts helps.]

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* The game is to pretend not to notice. Recipe: take a piece of dubious logic, then to make the 'conclusion' doubly sure raise it to the status of a universally acknowledged truth. Better still, start on a grand note by reversing the sequence. And *this* . . . is what Jane Austen does.

COMMENTARY: PART II

2 (c) WHY THE STUDY OF GEOMETRY IS SO VALUABLE

On seeing two women hurling insults at one another across an alleyway:

'Those two women will never agree; they are arguing from different premises'.

Sydney Smith

Traditionally, geometry is important for teaching the use of reasoning, and giving practice in it. And a context is needed in which to practice reasoning and communication, free of the influence of such things as beliefs, opinions, ideas and feelings. How these can interfere with communication is now discussed, and one suggestion given as to what can be done about it.

So far the discussion has focused on communication taking place fully, without a hitch, speaker and listener sharing the same thoughts [Diagram 1].

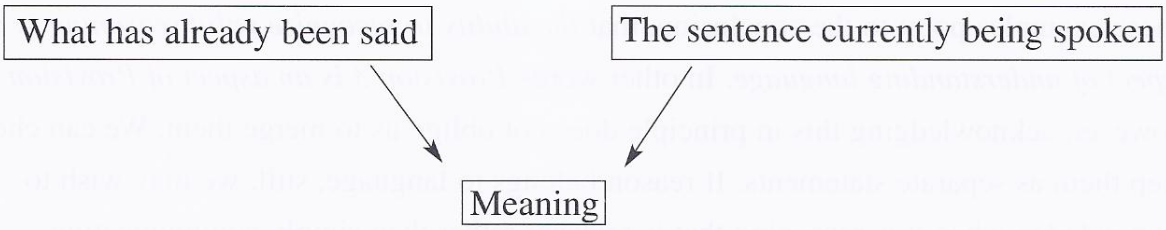


Diagram 1 Communication working fully

But we each of us have our own beliefs, opinions etc., and when something is said they can make an additional contribution to the meaning, so changing it. Some of them may act as though they are additional sentences, others may colour our understanding of what is said. They can be called *hidden factors* or influences. This is shown in Diagram 2.

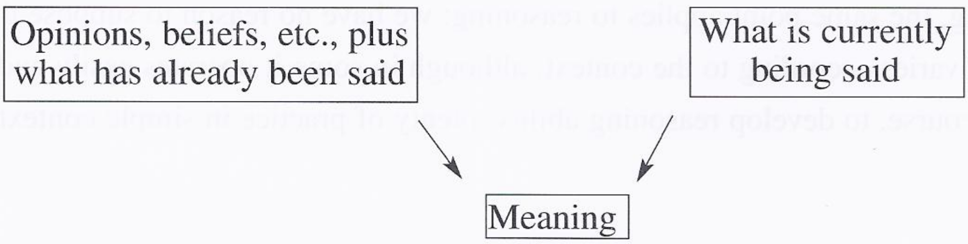


Diagram 2 Imperfect communication

However, the aforementioned points apply both to the speaker and to the listener, and a fuller version of the diagram needs to take this into account. [See Diagram 3].

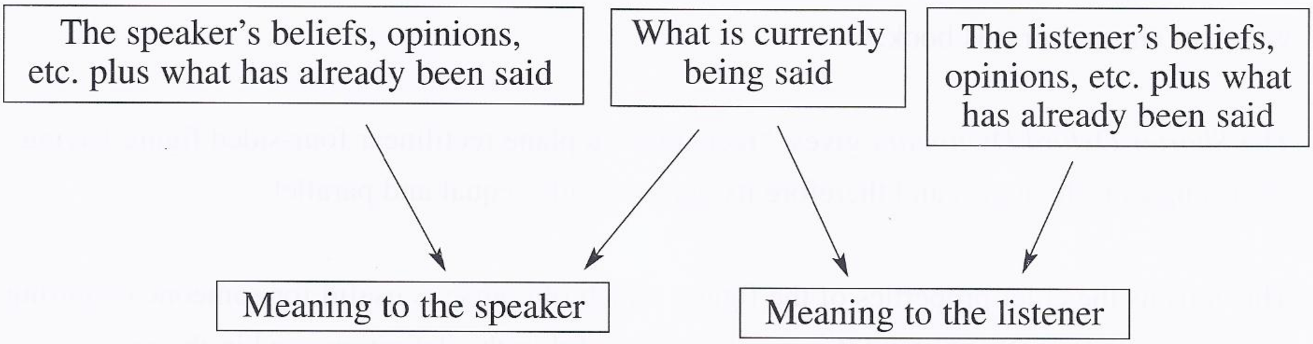


Diagram 3 Showing how misunderstanding commonly arises in communication (which may include an element of reasoning)

This is the situation. What can be done about it? One of the things which can be done is to practice reasoning in a context where hidden factors do not have a role. Geometry provides such a context.

We need to know what it is like to understand what is said free of hidden factors – not least, when there is an element of reasoning present. And one reason why we need some experience of operating under these conditions is so that we can get used to registering when communication and reasoning is not working in that way, but rather is influenced by such things as hidden assumptions.

3. THE VOCABULARY OF GEOMETRY

This essay discusses redefinitions: why words are redefined, and how words are selected for redefinition. It concludes with a discussion of the significance of the definitions of magnitude and equality. This shows that, a language with redefinitions being a storehouse of information, there is a choice between using it as such or finding alternatives (in the form of axioms).

3 (a) Redefinitions

Good dictionary definitions are available. Why do geometers insist on replacing them?

Briefly the answer is, to sharpen the language, adapting it to the purpose in hand.

As an example, consider the following comparison of a dictionary definition of a rectangle with that chosen for this book.

The *Shorter Oxford Dictionary* gives: "rectangle": a plane rectilinear four-sided figure having all its angles right angles and therefore its opposite sides equal and parallel.

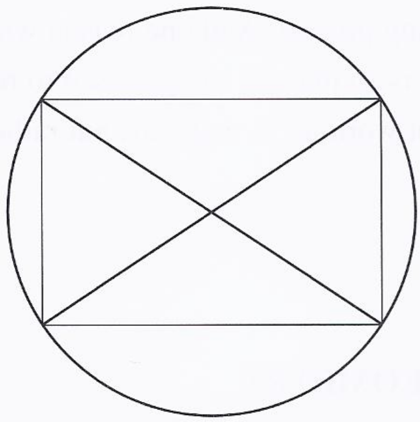
This tells us the chief properties of the figure, which of course is useful for someone enquiring what a rectangle is. By contrast it is much more useful in the definition used in the present text, to give just what suffices to specify the rectangle. And since it is already understood that the figure is in a plane, a single property of a rectangle suffices:

"rectangle": a quadrilateral with all angles equal.

This definition can be seen at work in the following construction of a rectangle.

Draw two intersecting lines, and a circle about the point of intersection (Fig 41).

Fig 41



The four intersections with the circle are the corners of a rectangle; for all four angles having been constructed identically, they are equal.

3(b) Other differences between dictionary definitions and those of geometers

Dictionary definitions commonly offer a number of meanings for a word, leaving it to the context to make clear the appropriate one. This provides possibilities of ambiguity, and there is nothing wrong with this; it is very useful for a poet. But geometers are made of sterner stuff and have their eyes on a different goal: unambiguity.

A second difference occurs because all dictionary definitions are given in terms of words, all of which are defined in the dictionary. In consequence, dictionary definitions are ultimately circular.

This can be put another way. Look up a dictionary definition. Noting one of the words in this definition, look it up in the dictionary in turn, and continue this process. Sooner or later a word will be encountered which has already been looked up. This gives the possibility of going round the same loop again, if we wish – or of finding another one.

In contrast to dictionary definitions, those of the geometer are given in sequence, just as steps of reasoning are given in sequence. The unwritten rule is, that no word which is to be defined may be used until it is defined. This tradition goes back to Euclid, possibly being even older.

Note that, in a list of definitions preceding a text, this rule is not a logical requirement. It can be breached without contravening the laws of reason – although, not being customary, it might be found disturbing. Yet words resting on dictionary definitions are subject to this sort of circularity, and we are obliged to make some use of them.

The linear sequence of the geometer may be likened to part of a straight line (a segment) and the circularity of dictionary definitions to a circle – two very appropriate images for a study of geometry.

The linear sequence of definitions works if it is short enough. An analogy is given by the surface of our planet, which can appear to be like a plane over short enough distances, but whose spherical nature asserts itself over longer distances.

3(c) A possible criterion for selecting the words to be redefined

During the nineteenth century the idea emerged that ideally, all words and phrases which refer to a figure in some way should be redefined. For brevity, let us call such terms 'geometric terms', or 'terms with geometric content'. An interesting thing about them is the way other geometric terms are used in their definitions.

Consider, for example, the first six definitions in Euclid's *Elements* (which begin his book):

Definitions

1. A POINT is that which has no *parts* or which has no *magnitude*.
2. A LINE is *length* without *breadth*.
3. THE EXTREMITIES OF A LINE are *points*.
4. A STRAIGHT LINE is that which lies evenly *between* its *extreme points*.
5. A SURFACE is that which has only *length* and *breadth*.
6. The EXTREMITIES OF A SURFACE are *lines*.

All six definitions use geometric terms to define geometric terms. Other examples show the same things happening. Why is this?

It can be argued, how can geometric content be brought into a definition, except by using terms with geometric content?

Let us consider the consequences of supposing that all geometric terms need other geometric terms in their definitions.

Consider all possible geometric terms, each with its own definition. Together they constitute a mini-dictionary. And they have the same circularity that occurs within a dictionary, for the same reason.

That is, all geometric terms cannot be defined without circularity – if all of them really do need to be defined using other geometric terms.

3(d) A rethink of the selection of words to be redefined

The evidence suggests that the proposed ideal is unattainable, and that all geometric terms cannot be redefined without circularity. To deal with this we could consider dropping the requirement that there be no circularity. But leaving that possibility aside, a response which comes to mind is to formulate a different ideal, making use of an alternative criterion to do so: words needed in the study are redefined, unless they are commonly used terms, in which case redefinition is optional.

This criterion accords well with an oral tradition. It has the consequence that the definitions depend on a core of words in common use. That is to say, we acknowledge that such words are well known and make use of them.

Note that the terms defined in this book can be put into four categories. These are, words related to: (i) drawing instruments and material; (ii) theorems, axioms, corollaries, etc.; (iii) figures; (iv) magnitude or its properties. The last two categories slightly overlap.

3(e) The role of definitions

Russell and Whitehead make some relevant observations concerning definitions in their *Principia Mathematica.*

(1) They say that theoretically it is unnecessary ever to give a definition: e.g. if a word is defined by a phrase, the latter can always be used, in which case the definition is redundant.

(2) They add "....nevertheless.......[definitions] often convey.......important information. This arises from two causes. First, a definition usually implies that the definiens [that which is defined] is worthy of careful consideration. Hence, the collection of definitions embodies our choice of subjects and our judgement as to what is most important. Secondly, when what is defined is (as often occurs) something familiar........, the definition contains an analysis of a common idea, and may therefore express a notable advance".

Two points conclude this section.

(1) Names of magnitudes apart, (length, angle, area), only two words concerning magnitude or its properties have been needed here: 'magnitude' and 'equality'. Being an important topic, this category is given a section to itself.

(2) Euclid's definitions largely relate to figures, except for his Book V definitions, which deal with properties of magnitudes. But he does not define either 'magnitude' or 'equality'.

3(f) The significance of the definitions of magnitude and equality

Firstly, the proofs given in this book and in Euclid's 'Elements' deal with the equality or lack of equality of lengths or angles or other magnitudes, and little else. Essentially, *Euclidean geometry is a study of magnitudes in figures.* If we omit to define key words in this fourth category, words important in proofs are left resting on dictionary definitions. [Except that Euclid has a different procedure. What follows explains this.]

Secondly, by defining magnitude it becomes possible to *demonstrate* that lengths and angles and areas are magnitudes, instead of just assuming that they are.

Thirdly, there is a significant issue concerning definitions and axioms, which will now be addressed. Consider the definition of magnitude.

Definition 4: Magnitude is that of which the whole in no way differs from a part save in size.

Here, the concept of size is understood. In consequence, we understand that some sizes are greater and some less. In particular, the whole of a magnitude is greater in size than a part.

This, which Euclid gives as an axiom, follows from our understanding the concept of size. More precisely, it follows on being granted the language and also Definition 7.

It can be expressed formally as a theorem, given here rather than in the text, since it has not been needed there.

Theorem Z The whole of a magnitude is greater than a part.

The above Proof of this theorem draws on our understanding of words. By expressing it as an axiom, Euclid is by-passing this procedure. He is not making full use of what is stored in the language and can be drawn from it with the aid of suitable redefinitions.
Objection: If a word such as 'size' is not assigned a special definition (i.e. redefined), surely it should not be assigned a key role in a proof.*

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* Examples of proofs resting purely on definitions are given in Propositions A10 and A12. In the latter, it is shown that the straight line through two points in a plane lie in that plane.

Answer: The position taken here is that the language is provided, including all its words, and that common or everyday words such as 'size' need not be redefined. This makes them available in proofs.

Consider next the definition of equality.

Definition 6: Magnitudes which are indistinguishable from one another except by position are said to be equal to one another.

This rests on the concept of indistinguishability. And it is the understanding of this concept which is used to prove Proposition A4:

Proposition A4 Theorem: Magnitudes which are equal to the same magnitude are equal to one another.

Euclid simply gives this as an axiom*- again by-passing what is stored in the language, and accessible with the aid of suitable redefinitions. The same remarks apply to Propositions A5 and A6, which Euclid also gives as axioms.

The conclusion is that drawing on what is stored in the language, with the aid of suitable redefinitions, can yield theorems which may otherwise need to be given as axioms, Otherwise put, axioms may be used to by-pass what is stored in the sharpened language (the language modified by the list of definitions).

One view of the matter is that we need not be overly concerned to make full use of what is available in the sharpened language, since it can always be replaced by axioms. A counterview is that it is more direct to make the best use of the starting point available with sharpened language, rather than bringing in axioms unnecessarily.

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* See first footnote, Part III

How would Euclid have reacted to all this? His chief concern would probably have been that the Proof of Proposition Z does not rest on any axioms or propositions, but on concepts contained within the language. In his system, although definitions do play a part, propositions are generally derived from other propositions or from axioms or postulates, which serve to set the ball rolling.*

In the present system, propositions are not only derived from other propositions, *but also from the Provisions*. And the language is included in the latter, giving it formal status (both as an instrument and as a source of material, in the form of words).

Thus in this system it is possible not merely to acknowledge that some words contain information of value in geometry, but to make use of them in proofs – whether they have been redefined or not.

However, it is also worth noting that it would be acceptable to treat the four theorems discussed here as axioms, as Euclid does. The validity of the present system does not depend on this issue.

3(g) The rationale of eight key definitions

Euclid's definitions serve as a useful comparison, for discussing the suitability of the definitions chosen here. The eight are: *angle, figure, line, straight line, surface, plane, circle, point*. For the first two of these the definition necessarily differs from Euclid's.

1. ANGLE The definition is, that *an angle is a measure of change in direction*. This works hand in glove with Prop. B4. An example is the theorem that the angles of a triangle total a half turn.

2. FIGURE Euclid defines a figure as *that which is enclosed by one or more boundaries*. This is too restricted in scope for the crucial Propositions A1 and A2 (q.v.). The much freer and more wide-ranging definition that *a figure is a drawing in a plane* meets their needs.

3. LINE How a line is defined, here, tells how one is drawn. The tip of the pen, idealized as a point, leaves a trace where it meets the paper. Hence the definition:
the path traced out by a point on a moving body is called a *line*.

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* However, in some of Euclid's proofs the object is to demonstrate that a certain definition is satisfied. For example, he shows that the construction of a square satisfies the definition of a square [*Euclid I , 46]*

COMMENTARY: PART II

In a much earlier draft I tried defining a line as *that which suffices to link two points*. But then there is the need to demonstrate elementary properties of a line, such as that its breadth is that of a single point (Prop. A8). These follow swiftly and easily with the definition chosen. Having resisted the latter initially, yet not seeing a better way forward, I was led to register that this is a satisfactory solution.

Euclid's definition of a line (No.2,p.64) is a good one, granted the meanings of *length* and *breadth*. Possibly it implies what is stated in Props. A8 and A9, and Corollaries, in which case it could be used here in this book. However, unless Euclid's definition of a surface is also used, his definition of a line looks out of place.

4. STRAIGHT LINE Euclid's brief definition (No.4,p.64) is a descriptive one, intended to leave us clear what a straight line is. He uses it by going on to grant that straight lines may be drawn (his Postulates 1 and 2). In practice an instrument is provided for this – the straight edge.

Yet given a straight edge, one needs to be given two points through which the line passes in order to draw it – unless any straight line will do. That being so it is more direct to use this in the definition:

> that line which is uniquely specified given two points on it
> is said to be *straight*.

Thus, both a line and a straight line are defined in accordance with the manner in which they are drawn (or would be, if we could draw the whole of a straight line).

SURFACES, INCLUDING PLANES

5. PLANE First, assuming that we know what a surface is, let us attend to the definition of a plane. Euclid's definition is ingenious:

> a plane surface is that in which any two points being taken,
> the straight line between them lies wholly in that surface.

The objection to this is that it includes an axiom to the effect that the straight line through any two points in the plane lies in that plane. [This is discussed in an Encyclopaedia article somewhere.]

Approaching the matter another way, we know from experience that it takes three points to support a plane. From this, the definition naturally arises, that *a plane is that surface which is uniquely specified given three points on it*. It follows that the straight line through any two points in the plane lies in that plane – no additional axiom needed here! (There is a benefit in so defining a plane that it is a generalisation of a straight line.)

6. SURFACE Besides using two points to position a straight edge, two things we want to be able to do when drawing in a plane are:
(i) to link two points by any number of lines, and
(ii) to draw a straight line through a given point in any direction of the compass.

By basing the definition of a *surface* on these two properties we save ourselves the trouble of proving that a *plane* possesses them. For it is (to be) granted that a plane surface is available. What follows carries out this plan.

As a first step towards this, note what a complete turn does: it passes through all the directions of the compass. This means that Item (ii) above can be replaced by:

> a complete turn can be made at any point of a plane surface,
> without leaving that surface.

This can be expressed more compactly:

> a plane surface contains a complete turn at each point.

However, a complete turn does not occupy an area. If we want any surface to be like a plane in the immediate locality of each of its points,* all we have to do is to stipulate that a surface contains a complete turn at each point.

These considerations give rise to the following definition of a surface:

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* I.e. not folded. Curved surfaces resemble a plane on a small enough scale, as we know from experience on our planet.

Definition A *surface* suffices to contain a complete turn at each of its points, and a multitude of lines linking any two of them.

NOTES (1) A line does not contain a complete turn, nor does a point. These may bound a surface but they do not themselves satisfy the definition of a surface. A line contains two directions at any point on it; a point contains none; a surface contains all the directions of a complete turn, at any point on itself.

(2) As defined, a surface may be of infinite extent, like a plane or an infinite cylinder: it may close in on itself like a sphere or doughnut (torus); it may have boundaries (lines) and/or holes (bounded by lines, and possibly by points also) and/or slits (consisting of one or more lines or points).

(3) A complete turn, considered as a whole, is like an element of a surface.

7. CIRCLE Before discussing the definition of a circle, there is a question needing attention, because it influences what is considered to be appropriate as a definition of a circle.

Question What is the distinction between a circle and its circumference?
Reply It seems the answer depends on which definition of a figure is accepted.

Euclid's definition A *figure* is that which is enclosed by one or more boundaries.

Accordingly, the circumference is the boundary, and that and everything inside it constitutes the circle. [Note that Euclid's above definition requires that figures be closed.*]

G.O.T. definition A *figure* is a drawing in a plane.

[**Alternative definition** A figure consists of lines and/or points in a plane. This alternative leaves no doubt that points are considered to be parts of a drawing.]

According to this, *the circle consists of the centre point and the circumference*. This answer influences the choice of definition of a circle [Definition 34].

* This can be put another way: Euclid's approach depends on it sufficing to consider closed figures.

Euclid's definition of a circle would be out of place here because it goes with a different definition of a figure. He speaks of a circle as being a figure *contained* by one line.

8. POINT Today, a point is thought of as a position. Add that it is without magnitude and a definition emerges. The definition chosen here uses the alternative statement that a point is without parts, from which it follows that it is without magnitude.

Definition A *point* has position but no parts.

However, there remains a question. Is such a definition a statement of what suffices to specify a point, or is it a full statement of its nature? Clearly it is not the latter. Then what is the full nature of a point?

PART III COMPARISONS WITH EUCLID'S ELEMENTS

Sir T. L. Heath's *The Thirteen Books of Euclid's Elements* is the translation used here.*

1. TESTS OF CONGRUENCE

EUCLID	GEOMETRY FOR AN ORAL TRADITION
The method of superposition	**Proposition A1 Theorem**
If one figure can be placed on another so that they coincide at all points the two figures are congruent.	Figures which can be constructed identically are congruent.

* Other translations differ over terminology – e.g. what Heath translates by *common notions* are called *axioms* in the Todhunter edition of the *Elements.*

COMMENTARY: PART III

2. SOME DIFFERENCES IN TERMINOLOGY

TERM	TRANSLATIONS OF EUCLID'S *ELEMENTS*	PRESENT SYSTEM
1 Straight line	Of finite length.	Unlimited in either direction.
2 Figure	Definition. A *figure* is that which is enclosed by one or more boundaries.	Definition. A drawing in a plane is called a *figure.*
3 Transversal	A line intersecting two parallel lines.	A line intersecting two other lines, whether the latter are parallel or not.
4 Corresponding angle	Used only if there is a line intersecting parallels.	Applies if there is a transversal to two lines, whether the latter intersect or are parallel.

3. EUCLID'S POSTULATES AND COMMON NOTIONS AND THE NEAREST EQUIVALENTS IN THE PRESENT SYSTEM

EUCLID'S POSTULATES	NEAREST EQUIVALENT
Let the following be postulated:	*Provision 2.*
1 To draw a straight line from any point to any point.	The following are provided: a plane, a pen, a straight edge and a pair of compasses.
2 To produce a finite straight line continuously in a straight line.	
3 To describe a circle with any centre and distance.	
4 That all right angles are equal to one another.	Prop. B7 All right angles are equal.
5 That if a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles.	Provision 4 Magnitudes are unchanged by motion. [This works through Proposition A3]

COMMENTARY: PART III

EUCLID'S COMMON NOTIONS	NEAREST EQUIVALENT
1. Things which are equal to the same thing are also equal to one another.	A4 Theorem Magnitudes which are equal to the same magnitude are equal to one another.
2. If equals be added to equals, the wholes are equal.	A5 Theorem Equals being added to equals, the totals are equal.
3. If equals be subtracted from equals, the remainders are equal.	A6 Theorem Equals being taken from equals, the resultants are equal.
4. Things which coincide with one another are equal to one another.*	A1 Theorem & A2 Theorem Figures which are or can be constructed identically are congruent, *and* magnitudes which are or can be constructed identically are equal.
5. The whole is greater than the part.**	Theorem Z The whole of a magnitude is greater than a part. [See Commentary, Part II, 3(f)]

4. POSTULATES, COMMON NOTIONS AND AXIOMS

"Aristotle says that every demonstrative science must start from indemonstrable principles: otherwise the steps of demonstration would be endless. Of these indemonstrable principles some are

(a) common to all sciences [the axioms]
(b) particular or peculiar to the particular science."

Sir T. L. Heath, *The Thirteen Books of Euclid's Elements.*

The evidence is that Euclid used 'common notions' in the same way that Aristotle used 'axioms'; Heath points out that Aristotle even describes axioms as 'common notions' at one point.

This suggests that Euclid's 'postulates' come under the category (b) above, which is in accordance with what seems to be the generally accepted view these days, that the postulates refer to statements granted relating to space.

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* This is a statement of the *method of superposition*. Today, *equality* of figures would be referred to as *congruence* of figures.

** This needs to be related to magnitudes, otherwise exceptions can be found. A well-known one is that 1+2+3+4+.... is not greater than 2+4+6+8+.... The two series can be put into one-to-one correspondence, showing that they contain the same number of terms.

5. EUCLID'S STARTING POINT SUMMARIZED

Euclid begins with what is presumably intended to be a résumé of what is known or granted or proposed at the outset. This provides the base from which the study is developed.

First, there are definitions proposed for parts of figures, and for the figures themselves. Then follow statements of certain things which are granted or assumed, those concerning figures being called 'postulates' (things which are postulated), and those of a more general nature being called 'common notions' (notions we already have, and which are used in the study). These form a base, from which the sequence of propositions unfolds, and on which the latter rest.

There are, in addition, certain unstated assumptions, such as that the rules of logic are available – or, at the very least, that valid reasoning can be recognized.

PART IV MOVEMENT IN GEOMETRY

1. THE WAYS IN WHICH MOVEMENT OCCURS IN *GEOMETRY FOR AN ORAL TRADITION.*

(1) Lengths and angles may be rotated or translated without change.

(2) Drawing a line entails movement of the pen.

The first of these comes under the Postulate, and the second is considered later.

2. WHY MOVEMENT IS INCORPORATED INTO THE STUDY

(1) *The criterion of congruence, Prop. A1, is crucial to the whole study*, and it rests on Provision 4 (the Postulate).

(2) *Provision 4 serves in place of Euclid's parallel postulate.*

If the latter is not to be used some alternative is needed. Over the centuries numerous alternatives were tried. Eventually it was found that it could be replaced by alternatives which contradicted it – and so non-Euclidean geometry was born.

Here, Provision 4 is used to show that intersecting straight lines make unequal corresponding angles with a transversal, and vice-versa. Hence the theorems on parallels.

(3) *Provision 4 permits the measurement of angles by rotation.*

Addition of adjacent angles is given in Def. 27. But by Provision 4:

(i) angles can be measured by rotation, and

(ii) angles are not changed on being moved from one place to another. Therefore non-adjacent angles can be summed by rotation, as is done in summing the angles of a triangle [Prop. B17]. This, incidentally offers another way of establishing the theorems on parallels.

(4) Note also that Euclid is unable to avoid including movement in his system, although he does so as surreptitiously as possible, placing the statement of the method of superposition amongst his Common Notions. If it is the case that movement cannot be avoided, why not include it openly? As the present study shows, doing so is highly productive.

3. AN ANCIENT AND A MODERN OBJECTION TO THE INCLUSION OF MOTION IN GEOMETRY

First, to the Pythagoreans, any hint of movement or change in geometry was anathema. In their view geometry dealt with a perfect world: changeless, eternally real. These were important considerations in Euclid's day.

Secondly, in modern times attention has focused on another reason for objecting to the introduction of motion into geometry. And that is the possibility that movement changes properties such as length. The advent of non-Euclidean geometry increased awareness of this possibility, and relativity theory further enhanced it. For one of the characteristics of its geometry is that movement *does* affect our judgement of lengths.

4. OBJECTIONS ANSWERED

FIRST OBJECTION

It can be argued that there is no need to deal with this one, since it belongs to an ancient and outmoded way of thinking. And indeed, the decision has been taken here not to be inhibited by it. This suffices as an answer. However, one relevant aspect of the world-view of those times is discussed shortly, in Section 5: it provides the context, both for this first objection to movement in geometry, and for the formulation of the *Elements*.

SECOND OBJECTION

(i) *Role of the Postulate*

Agreed, it is possible that movement changes such things as lengths and angles. However, it is also possible that movement does *not* change them, and it is this second possibility which is explored here.

For different systems of geometry are possible, having different formulations. It is for us to choose a formulation, and the choice made here is of one satisfying the Postulate. That is, the system formulated is one which is unaffected by movement.

(ii) *Stated and Unstated assumptions*

If assumptions or agreements are stated they become part of the formulation: it gives them status. Unstated agreements, by contrast, creep in by the back door. They may be more or less acceptable but they have not been stated as part of the set-up.

The objection that motion might change length (or other magnitudes) carries weight as an objection to an unstated assumption. Here, however, motion has been incorporated in the set-up in the form of the Postulate: it is a stated assumption. In this way the consequence is explored of supposing that movement does not change magnitudes.

In contrast, if motion is included as an unstated assumption, the objection holds, that motion might change length etc., and that we are not entitled simply to assume that it does not. This objection can be levelled against Euclid's use of the method of superposition.

5. A FURTHER OBJECTION AND HOW IT IS ANSWERED

Bertrand Russell raises an objection to Euclid's method of superposition which is relevant also to Provision 4. He argues that motion is only possible for bodies, not space:

"The motion of a point of space is a phantom directly contradictory to the law of identity: it is the supposition that a given point can be now one point and now another. Hence motion in the ordinary sense is only possible to matter not to space." [*The Principles of Mathematics*]

If it is not acceptable to speak of moving points, on the same grounds it is not acceptable to speak of moving lines – which would mean that a line is now this line and now that one. And the same objection applies to a moving magnitude: now it is this magnitude, now another one. It seems that the argument undermines the Postulate, which uses moving magnitudes.

To answer this, let us take a fresh look at where our ideas of rest and motion arise. *Moving* and *stationary* are abstractions from our experience, e.g. of walking about on the surface of the earth, and moving on it or on the sea, in vessels and vehicles.

Abstraction of ideas is natural to us, and essential in the use of language. For example, a flat portion of the earth, and flat surfaces on buildings and furniture, have something in common. From this we get the idea of a plane.

Our experience of movement can lead us to the idea that movement is relative. Travel in a boat or on a sledge or in a car gives us experience of a moving frame of reference, within which things can be still. But because the earth is so much more steady and reliable as a frame of reference, as well as being on a larger scale, there is a tendency to extrapolate, and consider that one reference-frame is stationary, and others are moving relative to it.

COMMENTARY: PART IV

When Euclid was formulating the *Elements*, prevailing opinion was that the earth was stationary, and that the sun and moon and stars moved relatively to it. In this context an absolute distinction between motion and rest makes sense. Euclid would not have needed to mention this background to his studies, because it would have been widely understood. In consequence, the assumption that a plane is granted is not mentioned by Euclid. Nor, so far as I am aware, does it seem to have occurred to commentators to mention it subsequently. Something everyone understands can easily not be mentioned.*

Nor does Euclid mention motion, yet he provides for it. It is there but not fully acknowledged, a sort of halfway house between a stated and an unstated assumption. His Common Notion 4 contains a statement of the method of superposition, omitting acknowledgement that movement is involved. The latter is something tolerated because it is thought to be necessary, but not stated because the official view of the Pythagoreans bars motion from geometry.

In consequence of the lack of acknowledgement that a plane is granted, or that movement is grudgingly admitted into Euclid's system, when Bertrand Russell discusses Euclid's method of superposition he does not question the assumptions about rest and movement underlying Euclid's approach. For it is implicit that rest is absolute, and movement is relative to that.

It is of interest to us then, to go back to our experience of the world, from which ideas about movement originate. And there, in experience, is to be found the principle that motion is relative. If I am moving relatively to you, then conversely you are moving relatively to me. The notion of absolute rest made sense when the earth was thought to be at the centre of the universe, but not now.** Replacing the idea of absolute rest by the principle that all motion is relative, rest is to be found in any reference-frame.

Re-examining Russell's argument in this light is instructive. The objection he raises to 'moving' bodies applies equally to the 'stationary' system. Has he come up with an argument to show that *any* frame of reference is impossible, even a 'stationary' one?

To repeat, if there are difficulties with associating a reference-frame with a 'moving' body, the same difficulties apply to associating one with a 'stationary' body. But this is a process of

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* This is an interesting thing about such common assumptions. They have to be inferred because no-one mentions them. Yet they are an important element in the thinking of the time.

** Copernicus, placing the sun at the centre instead, can be seen to have related *absolute rest* to the *sun*, removing the former from immediate experience. In retrospect, this was a step towards abandonment of the very notion of absolute rest.

abstraction from experience, which involves looking to an ideal, and for this we look beyond the difficulties.* The abstraction is a sort of limiting process. In selecting a reference-frame for earth we do not go through the sort of difficulties Russell mentions, simply because our minds reach beyond them, in formulating the ideal of a reference-frame. It is true however, that on examining our abstractions later, we may find that there are difficulties associated with them.

And here also is the answer to another question: *what is meant by a 'moving' magnitude*? The answer is that it is *a magnitude which is at rest in some 'moving' reference-frame*.** Underlying Provision 4 is the the principle that motion is relative. Unlike Euclid's *Elements*, this system does not embody the notion that there is an absolute state of rest.

6. THE OBJECTIONS AND EUCLID'S *ELEMENTS*

It is of interest to explore further how the two objections of Section 4 apply to Euclid's *Elements*, partly by way of contrast and partly because of the important role played by the *Elements* in the development of the Western intellectual tradition.

Continuing with the second objection, note that the method of superposition may well have been a standard method in Euclid's day, which might be one reason why he gave it as a Common Notion but did not otherwise discuss it. However, he used it as little as possible – and has been criticized for not using it more, since it would have meant less work. This under-use of the method suggests that he disliked it, movement being implied. Furthermore, he very carefully avoids direct transmission of length using compasses (other than in drawing a circle) – a scruple which also suggests he did not wish to include movement if he could avoid it.

Let us now consider the influence on Euclid of the first objection, the Pythagorean view that geometry deals with changeless reality, and that change or movement has no part in it.

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* It seems that what happens is that abstractions arise from our experience of the world, and later we discover that there are problems associated with them.

** It can be argued that this makes the earlier comments out of date – the strictures on speaking about 'moving points' and 'moving lines'. Yet these serve as useful reminders not to take these things for granted. A *'moving point' is in fact stationery, on some body or in some reference-frame.* That is what is important about relating it to its appropriate reference-body or reference-frame.

Debarring change from geometry meant that an angle could not be defined as a change of any kind, let alone a change in direction. Even the concept of direction was suspect, a direction being, *from* somewhere and *to* somewhere else.

Hence the definition of angle used here, and the Postulate, were both out of bounds to Euclid. He was therefore obliged to find other ways of proceeding.

As the present text shows, this was at the cost of brevity – not that brevity seems to have been of especial importance to Euclid. It appears that his interest was more in providing a chain of reasoning proceeding from first principles to the construction and properties of the five Platonic solids.

To conclude the present discussion, Pythagorean ideas had considerable merits. They were of great beauty and charm; they inspired thinkers, and provided a framework within which to proceed; yet they did constrict thought. Ultimately they were superseded; yet they left behind a legacy, not least through Euclid's *Elements*.

7. PROOF OF THE METHOD OF SUPERPOSITION AS USED BY EUCLID

A triangle is bounded by three segments. Given that segments are unchanged by motion, does it follow that triangles are unchanged by motion?

Not necessarily; for conceivably the angles of the triangle may be changed; also, the area of the triangle may be changed. But being given the Postulate, that all these magnitudes are unchanged by motion, there is nothing to distinguish the triangle in different positions or different states of motion. That is, the triangle is congruent with itself, whatever its position or state of motion.

Thus the Postulate can be used to demonstrate Euclid's method of superposition as applied to triangles – which is all he uses it for. But it is more effective to use the Postulate to demonstrate Proposition A1, that identically constructed figures are congruent.

To complete this Section one thing remains to be discussed – the second of the items on the opening list, of ways in which movement enters this study of geometry.

8. THE MOVEMENT OF THE PEN IN DRAWING A LINE

Why define a line as the trace of a point on a moving body?
Because: (i) this definition is in accordance with the way a line is drawn, using a pen, and
(ii) it follows from this definition that a line is one point thick.
Lines are elements of a surface (Definition 21), but they are not visible until drawn. Thus, drawing a line is making visible something that is already there. But if so, how can drawing a line change what is there?

Yet a counter-argument is, how do we know which line we are revealing by drawing one? Is the line being drawn the one we think it is? This is where the provision of drawing instruments to guide the pen comes in. Being granted a straight edge, we are granted the straightness of lines drawn with its aid. We know that we are drawing a segment.

Here we are beginning to stray into the next topic, the validity of figures. What comes now prepares the way.

TWO BASIC QUESTIONS ASKED BY THE ANCIENTS

These are included here because they help to set the scene for the next topic. In ancient Greek thought two questions which might be asked of something were:

(i) does it exist ?

(ii) what is its nature, i.e., what are its characteristics?

In geometry this led to propositions being of two types:
the problem, concerned with establishing the existence of a figure, which meant showing it could be constructed, and
the theorem, concerned with stating something about a figure.
Proclus puts it in this way, in his commentary on Euclid's 'Elements':

> "Hence too Posidonius defined the one (the problem) as a proposition in which it is inquired whether a thing exists or not, the other (the theorem) as a proposition in which it is inquired what (a thing) is or of what nature."

PART V THE VALID USE OF FIGURES

1. PREAMBLE

Examples exist of fallacious arguments, yet we are not to conclude from this that reasoning cannot be trusted. After all, rules are available governing valid reasoning – the rules of logic. Likewise, in geometry there are fallacious proofs of theorems using figures which are not what they are supposed to be. Here too, it does not follow that figures cannot be trusted. Indeed, *the study of geometry rests on the premises that figures possess very definite and reliable properties, and that we can know them.* If that is so, there will be rules governing their valid use, even if we do not know what they are.

Some of these rules have been known since ancient Greek times. But in the nineteenth century questions were asked about the use of figures which culminated in systems of geometry being developed which withdrew from using figures altogether.

That is one way of setting about dealing with problems arising in using figures. Another is:

Firstly: to acknowledge that rules governing the valid use of figures exist (for if not, the above-mentioned premises on which the study of geometry rests cannot both be valid).

Secondly: to note those rules which are known.

Thirdly: to seek to formulate those of the rules which appear to have eluded us so far.

The issue is relevant here, for the present study uses figures in proofs. How do we know that they are valid?

We begin with clear cut rules, known and put into practice at least since Euclid's time. Then follow less clear cut rules, put into practice to some extent from Euclid onwards, and tentatively formulated here. Finally, some of the questions raised by nineteenth century geometers are considered briefly. Noting what appears to have been their chief concern suggests an addition to our proposed list of rules.

2. RULES GOVERNING THE VALIDITY OF FIGURES

1. The validity of each figure is established by giving a construction and proving that the figure so constructed satisfies its definition or description.
2. Different cases of the figure need to be acknowledged, and proofs given to cover each case.

Between them these rules contain a statement of what a problem of construction sets out to achieve. For example, to demonstrate that a particular figure is a square, we show that it is so constructed as to satisfy the definition of a square. They cover more than this, for they apply to the figures used in theorems, not just problems. Sometimes we are dealing with figures which have been defined, sometimes they are given by description. These need to be considered separately, in applying them to the present text.

3. DEFINED FIGURES

Being given a definition of a figure does not establish that it exists. Leibnitz gives the example of a ten-faced regular solid, which can be defined but not constructed: no such solid exists.

There are a number of defined figures whose existence needs to be established by giving a construction and proving it. These include a point, segment, circle, triangle, quadrilateral, rectangle etc. (Bear in mind that, as defined here, any configuration of lines and/or points in a plane constitutes a figure).

The provision of a straight edge and compasses (and a pen) establishes that segments and circles can be drawn. Again, by definition, points (and lines) are elements of a surface. Therefore, granting a plane surface establishes that points exist.

For other figures a construction is required. The following table sets out the requirements in a few simple cases.

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* Likewise, lines are granted to exist; drawing one merely reveals it. But if all lines and points already exist in the plane, so do all figures; drawing them reveals them.

COMMENTARY: PART V

FIGURE	HOW EXISTENCE IS ESTABLISHED
1. Point	**Granted [Provision 2]**
2. Segment	**Provision of a straight edge [Provision 2]**
3. Circle	**Provision of compasses [Provision 2]**
4. Triangle	**From a common point, draw two segments, not in the same straight line. Join their free ends. This figure satisfies the definition.**
5. Isosceles triangle	**Drawing a circle, use two of its radii as two sides of a triangle, which is then an isosceles triangle.**
6. Cyclic-quadrilateral	**Place four points on a circle. Join neighbouring points to form a quadrilateral on the circle**

There follow a number of points concerning the above matters.

a) The reader may be interested in what Aristotle has to say concerning this issue:

> "....we have to assume the existence of a few primary things which are defined, viz., points and lines only; the existence of everything else, e.g. the various figures made up of these, as triangles, squares, tangents and their properties needs to be proved."
>
> [*Posterior Analytics]*

b) Some figures cannot be drawn fully: e.g. a straight line, and a pair of parallel straight lines. In these cases we have to be content with part of the figure.

c) Again, some figures cannot be drawn directly from the definition. Examples are parallels (straight lines which never meet) and a tangent to a circle. That a tangent meets a circle once does not suffice to draw it; that it is perpendicular to a diameter does.

d) There is also the question, since magnitudes are parts of figures, does their existence need to be established in the same way? Is the existence of angles demonstrated by their existence in figures such as triangles?

e) In the present book, the existence of figures is not systematically established first, before using them. It seems to be a better idea to defer the issue, and then mentally insert the relevant steps, where needed, on going through the material again.

f) Finally, how does Euclid deal with this issue? Aristotle's thoughts on it would have been available to him. Yet this is not the only reason for believing he was aware of the matter. As Sir T. L. Heath points out, not until he has given a construction of a square does he use the square in theorems. However, with more elementary figures, such as triangles, including isosceles triangles, he is less rigorous, assuming their existence instead of demonstrating it first – although he does start Book I with the construction of an equilateral triangle.

4. FIGURES GIVEN BY DESCRIPTION

Theorems about figures need to refer to the relevant figure in some way. Often this is done by describing it briefly.

Example, Proposition D1 Theorem. A diameter of a circle subtends a right angle on the circumference:

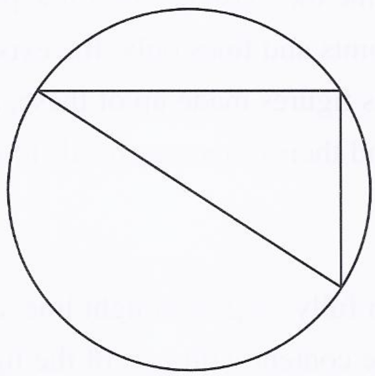

Fig 42

As with defined figures, the requirements for described figures are that:

(i) a construction be given, and

(ii) the figure so constructed accords with its description.

These are often the first two steps of a proof. They can be observed in D1 Theorem. Firstly, the construction draws on B22 Problem, the construction of a rectangle. Secondly, observation shows that Figure 29 satisfies its description.

Sometimes the description of a figure consists of the steps by which it is constructed. This occurs, for example, where further construction is required for the proof. To establish that the figure is what it is said to be, in such a case, it suffices to carry out the steps of the construction.

5. A FURTHER ISSUE: THE VALID USE OF FIGURES

As well as the requirement that figures are valid in themselves, there is a need to discuss them. This brings in the use of language: words are needed to refer to figures. We are dealing with the meeting of two systems, that of figures and that of language. Rules are available governing the valid use of language (the rules of logic), and likewise rules are available governing the validity of figures. What is needed now is to formulate how figures and language relate to one another. If we succeed in formulating that perhaps we shall have achieved our objective: a formulation of rules governing the valid use of figures.

First, we need a clear statement of the definitions applying to all geometric terms (words relating to figures). For this is where the two systems meet. In the present book, for example, many of these geometric terms are redefined at the beginning of the text, and all the others used – being common or everyday words – rest on dictionary definitions.

A little historical background helps to lead up to the next point. In the nineteenth century, in the wake of the emergence of non-Euclidean geometries, there was a questioning of traditional assumptions in geometry. I believe it was the Italian school of geometers who suggested that it is desirable not just to assume that we know what is meant by words such as 'between' and 'inside', but to furnish mathematical definitions for them. However, as was pointed out earlier in the Commentary, it appears we cannot redefine all geometric terms without circularity, so that some selection among them is required (if circularity is to be avoided).

Such matters can be approached in more than one way, but we do need to be clear which approach we are following. That is to say, *figures are being considered in the context of a system, or approach. What that approach is needs to be stated.*

Furthermore, those aspects of the approach which have a bearing on the definitions in use need to be stated.

Of these, the chief is surely the principle followed in selecting words for redefinition. Such a principle gives a sense of order to the study, and aids in appreciating the system as a whole – which makes for its intelligent use. Without such a principle there is a risk of the study appearing to be haphazard – not a quality to be welcomed in what purports to be a systematic study of geometry. As already stated, in the present book, the principle is that words in common use need not be redefined, but others are.

It is possible that it is also relevant to state any assumptions or agreements concerning the use of language. For any restrictions or conditions on the language may have a bearing on how it relates to figures. And if not, it is helpful to be clear that they do not. In the present text, the relevant points are that a language is granted, and that dictionary definitions apply until replaced.

To summarize, the rules suggested so far governing the valid use of figures are these:

1. That a figure is valid in itself needs to be ensured, and

2. different cases of the figure need to be taken into account.

3. The definitions of terms used relating to figures need to be made available.

4. The criterion or principle by which words are selected for redefinition needs to be clear.

5. It needs to be clear which system or approach is being followed (to put the study in context).

6. Any other features of the system which have or may have a bearing on terms used relating to figures need to be stated: e.g. any assumptions or agreements concerning the use of language.

But this list of rules is not complete, as the following considerations show. Euclid has been accused of excessive reliance on figures. A figure used in a proof is intended to represent a general case. For example, 'the angle subtended by a chord on the circumference' is not limited to an angle subtended in one place, although the figure only shows one of the possibilities. An attempt to take this into account in proofs is made by considering different cases. The latter arise, especially, where some further construction is added to the basic figure [for example, see Proposition D2]. Yet *the balance between what can safely be taken from the figure and what needs to be proved is not always clear cut.* Some of Euclid's decisions, in this respect, have been challenged.

Along with this consideration, there is another, not unrelated one. Euclid does not give definitions for terms such as 'between' and 'inside', which describe geometric relations. It can be argued that this is unsatisfactory. What these issues have in common is their topological nature, it seems. This suggests that what is needed is an additional rule:

unstated assumptions of a topological nature are unacceptable in proofs.

For stated assumptions are taken into account in proofs. It is the unstated ones which need to be watched. If this additional rule is found to hold good, and to be of practical use, we then have the following rules:

Rules governing the valid use of figures

1. Each figure needs to satisfy its definition or description.

2. The different cases of the figure need to be taken into account in proofs.

3. Unstated topological assumptions are not acceptable in a proof.

4. The list of redefinitions of terms related to figures needs to be given.

5. The criterion or pattern by which terms are selected for redefinition needs to be stated.

6. The system or approach being followed needs to be stated.

7. Any assumptions or agreements made concerning the use of language need to be stated.

This is, of course, a provisional list – possibly no more than a stage in the search for the appropriate rules.

The third of these proposed rules looks as though it could lead to considerable further investigation, although this is not being followed up here. For example, questions which come to mind include:

What forms can statements of topological assumptions take?
Can a figure be a statement of topological assumptions?
If so, under what circumstances?

In conclusion, this investigation is far from complete. Yet enough has been said to make the point that not only are such rules needed, but we have an idea of what some of them are.

6. GEOMETRY WITHOUT FIGURES

It must be possible to replace in all geometric statements the words *point, line, plane,* by *table, chair, mug.* D. Hilbert

In his book, *Foundations of Geometry*, (first edition 1899), Hilbert formulated a system of geometry in which three terms were left totally undefined, that is, not even covered by dictionary definitions. Among other considerations, this was one way of resolving the problems encountered on attempting to define all terms related to figures.

In addition, he set out to formulate our intuitive understanding of figures in a set of axioms. But if the essential information in figures can be axiomatized, do we really need the figures themselves? Hilbert took the bold step of dispensing with them. And since he left 'point', 'line' and 'plane' as undefined terms, how could figures be used anyway?

Thus two leading characteristics of Hilbert's formulation are:

(1) 'Point', 'line' and 'plane' are at no stage defined.

(2) A set of axioms is used which makes it possible to dispense with figures.

Because of (1), axioms frequently serve in place of definitions. For what sense are we to make of a definition which depends on an undefined term? But an axiom, used instead, may tell us some useful feature of the system – much as a definition may refer to some distinguishing feature.

In consequence, the approach uses a goodly number of axioms, but few definitions. This contrasts with the present book, which uses definitions freely, but few provisions.

The system is developed in such a way that although in principle figures are not relied upon, they can always be used by way of illustration. To do so, 'point' '(straight) line' and 'plane' are given their ordinary meanings. But *that such figures are used validly does need to be established.* The rules governing the valid use of figures have still to be satisfied. And this, even in a system which seeks to free itself from dependence upon figures!

Hilbert's system encountered opposition at first, but gradually gained acceptance. And there have been a number of formulations on much the same lines since. In consequence, there is a modern approach to geometry in which the role of figures is pushed into the background.

Along with this development, expressions of distrust in figures are not uncommon. This is odd, because *essentially, each of the various branches of geometry is a study of the properties of figures* (if not, what right has it to be considered a branch of geometry?)

If figures themselves cannot be trusted, what trust can be placed in a study of their properties?

As an example, here is an extract from what C. S. Ogilvey has to say about figures (diagrams) in the introduction to his book, *Excursions in Geometry*:

> One must guard against thinking, "The diagram proves it." Appearances are often misleading: diagrams are useful only as an aid to picturing things that can (at least theoretically) be stated *and proved* without them. Yet they are so useful in clarifying our thinking that only the most abstract purists attempt to dispense with them entirely.

This distrust of figures points to something being unsatisfactory. It seems the problem is lack of general acceptance that reasoning with figures can be valid if pursued lawfully.

Even if this is accepted in principle, there is still a need to state the relevant rules. The formulation suggested here is an initial attempt to do so, presented for discussion.

SUMMARY AND CONCLUSIONS

A sequence of propositions has been provided, establishing the elementary properties of a circle. It is based on four provisions.

Euclid's five Common Notions and five Postulates are shown to have approximate equivalents here – sometimes close ones, sometimes meeting the need in a different way.

Evidently what is presented here is part of a system, which is ripe for further development.

Chief innovations

1) Movement has been objected to in geometry in both ancient and modern times. Yet it lies at the centre of this system, embodied in a postulate. Being a stated assumption the drawbacks of unstated assumptions (of motion) do not apply.

2) A need is recognized: to state the assumptions knowingly made. This results in the concept of *Provisions*, things provided at the outset, including any axioms or postulates, the language used etc.

3) The need for rules governing the valid use of figures is pointed out, and the process of formulating such rules is begun.

Other points

1) The language is central to an oral tradition. It is developed in such a way as to provide for studies such as that of geometry.

2) One of the Provisions being language, proofs can rest on the meaning of words, thereby reducing the number of axioms needed.

3) Grounds are shown for believing that it is not possible to redefine all terms relating to figures without circularity. Therefore another criterion is selected for choosing words to be defined: words needed in the study are redefined, except that this is optional for words in common use.

4) The Postulate rests on the principle that motion is relative: each 'moving' magnitude is at rest in some reference-frame.

5) It is shown that in order to understand speech we sometimes need to draw on the ability to recognize valid reasoning, and that this occurs more widely than is commonly recognized. In consequence, Provision 3 is seen to be part, albeit a special part, of Provision 1.

To conclude, having finished this book I am left with a sense of potential. It is as though, having become aware of a gold-mine, what has been done but scratches the surface.

Questions

1. What is the equivalent of a dictionary in an oral tradition?
2. In what ways is mathematics notation of use in an oral tradition?
3. Can a figure be a statement of topological assumptions? If so, in what way?
4. It has been noted (in the Preliminaries) that concepts have implications. How do we know that the concepts represented by words in common use are consistent with one another? This is not limited to the concepts of geometry: it has a wider application. Furthermore the question can usefully be related to the whole of language, not just words in common use. But in the present system the latter are especially important because they constitute a core of words on which the study is based.

Questions and some answers

Question 1

Is it more apt to speak of a figure *in* a surface or a figure *on* a surface?

Answer

Both are equally acceptable. The nature of a surface is such that a figure lying in it also lies on it, and vice-versa.

Question 2

Are different words needed for congruence and equality?

Today we speak of congruence with reference to the sameness of figures, and of equality when dealing with numbers or algebraic expressions. The ancient Greeks did not draw this distinction. The advantages and drawbacks of the ancient and modern approaches to this might make a good topic for discussion.

Question 3

What is geometry?

Answer

Geometry is the study of properties of figures. Euclidean geometry is a study of magnitudes in figures. It examines, for example, the equality or lack of it, of lengths and of angles and of other magnitudes.

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APPENDIX 1

APPLICATION OF THE SIXTEEN SUTRAS TO THE PRESENT SYSTEM OF GEOMETRY

Tirthaji's sixteen sutras are like patterns or themes. Nine are used in the following study. But first, let us consider, what would be a good procedure for investigating the application of the sixteen sutras here?

Perhaps the ideal is to begin by seeking to find out how people thought in those times before writing became established. For instance, if we were studying aspects of 14th century French thought, a key question would be, how did they think? And we would do well to study original writings of the times, in the original language, to find out what factors influenced them. In the present case, the earliest writings would surely have coincided with and reflected a continuing oral tradition, even though the latter was by then doubtless in decline. Indeed, it may be surmised that only in its decline would the thought of writing down the oral tradition have been taken seriously, this being contrary to its spirit.

A further consideration is that within a given culture there is variation: for instance, each academic discipline can be expected to have its own ways of thinking. But there is little or no information on the mathematics of those times. Complicating the matter, eastern writings have not been documented anything like as fully as western ones.

Tirthaji must have followed something like this approach himself, and it may well be feasible for others to do the same, although it is not straightforward. However, there is another and very different way of tackling the issue. Suppose we *assume* that Tirthaji is correct, and that mathematics in its entirety comes under the sixteen sutras he has formulated. The job then becomes to make sense of this statement, in the context of the present system of geometry. In short, the task is to find *how* the sutras apply to the present material, the assumption being that they do. That is the approach being taken here, and in following it, it helps to be willing to be flexible in interpreting and using the sutras – willing to stretch the meanings a little, willing to consider the possibility that in places a single translation of a sutra from the Sanskrit may be insufficient, even though we do have Tirthaji's own translation for most of them.
NOTE: The reader is advised to have a book-mark handy for easy access to the relevant Proposition (in Parts A- D).

APPENDIX 1

Applying the sutras

HYPOTHESIS: the sutras apply to each and every part of mathematics.

PART A

A1 Theorem

SUTRA: Paravartya yojayet, TRANSPOSE AND APPLY (Tirthaji's translation).
The sutra may be slipped into a sentence describing (or restating) the Theorem, thus: *transpose* the place, *and apply* the same rules of construction, and the result is the same. The same sutra also applies to Euclid's method of superposition, which is his basic method for testing congruence (see Commentary).

A2 Theorem, A3 Theorem and A4 Theorem

These use the same sutra as A1 Theorem.

A5 Theorem and A6 Theorem

SUTRA: Sankalanavyavakalanabhyam, BY ADDITION AND SUBTRACTION (Tirthaji's translation).
This pair of theorems may be jointly formulated, incorporating the sutra: both *by addition* of equals, as in A5 Theorem, *and by subtraction* of equals, as in A6 Theorem, the results are equal.

A7 Theorem

SUTRA: Shunyam Samyasamuccaye, IF THE SAMUCCAYE IS THE SAME, THAT SAMUCCAYE IS ZERO (Tirthaji's translation).
This English translation can be shortened:
IF THE SAMUCCAYE IS THE SAME IT IS ZERO.
We will not be misled if we regard 'samuccaye' as referring to 'something', the interest being in something which remains the same. 'Samuccaye' can refer to a single term, such as a magnitude, or to some combination of terms (see 'Note on Samuccaye' below). On page 107 of *Vedic Mathematics* Tirthaji gives an example in which 'the samuccaye' is the same but something else is zero. This stretches his own translation beyond normal usage of the English – and it suggests alternative translations:

APPENDIX 1

IF THE SAMUCCAYE IS THE SAME THERE IS THAT WHICH IS ZERO, OR,
IF THE SAMUCCAYE IS THE SAME THERE IS A ZERO.

Turning now to A7 Theorem, having no parts a point is *the same* throughout itself – and the *zero* is its magnitude.

NOTE ON SAMUCCAYE

For the various meanings that Tirthaji assigns to 'samuccaye' see *Vedic Mathematics*, pages 107 *et seq.* Two Sanskrit – English dictionaries give:

Samuccaye = aggregate, accumulation, collection, assemblage, multitude, totality, aggregate. (Monier – Williams, page 1165 column 2)

and

Sam-uk-kaya = heaping up together, mass, multitude, totality, aggregate. (McDonnell, page 338, column 2).

A8 Theorem

SUTRA: Shunyam samyasamuccaye, IF THE SAMUCCAYE IS THE SAME THERE IS A ZERO.

Here, the line is *the same* throughout itself, and the *zero* is its thickness.

A8 Corollary

SUTRA 1: IF THE SAMUCCAYE IS THE SAME THERE IS A ZERO.

Two intersecting lines coincide (are *the same*) at a single point, which has *zero* magnitude.

SUTRA 2: Sopantyadvamantyam, THE ULTIMATE AND TWICE THE PENULTIMATE (Tirthaji's translation).

The word 'penultimate' comes from the Latin, *paene*, meaning 'almost' and *ultimus*, meaning 'last'. Noting this, and that Cassell's dictionary gives 'penultimate' as meaning 'next to the last', the following alternative translation for the sutra is proposed:

THE ULTIMATE AND TWO PENULTIMATES.

APPENDIX 1

For it is not uncommon for there to be more than one term next to the ultimate – 'the ultimate' being whatever is considered to be the last or ultimate term, or the objective or goal currently aimed at. To some extent, use of the terms 'ultimate' and 'penultimate' may just be a convenient convention for the purpose in hand.

Variations on this alternative translation include:
THE ULTIMATE AND THE PENULTIMATE PAIR, and:
THE ULTIMATE AND DOUBLE PENULTIMATE.
This last form can be interpreted as incorporating both meanings – the one given by Tirthaji and the proposed alternative.

Relating this discussion to A8 Corollary, the two lines can be seen as the pair of penultimates which lead to the goal, or ultimate – i.e. the point of intersection.

A9 Theorem and A10 Theorem
SUTRA: THE ULTIMATE AND TWO PENULTIMATES
One type of application of this sutra is where two like things or qualities come together, yielding something else. In A8 Corollary, two (intersecting) lines yield a point. And that two points specify a straight line (Definition 12) is the key to A9 Theorem and its Corollary and A10 Theorem. The sutra draws attention to a pattern that these Theorems and A8 Corollary have in common – one of interest in topology.

A11 Theorem
This is discussed after Part D, along with the definition of magnitude.

A12 Theorem
SUTRA: Shunyam samyasamuccaye, IF THE SAMUCCAYE IS THE SAME THERE IS THAT WHICH IS ZERO.
In this case it seems that 'the samuccaye' refers to the plane and the straight line – or perhaps to what they have in common – and the zero to the extent that the straight line departs from the plane.

A13 Theorem and A14 Corollary

SUTRA: Sopantyadvayamantyam, THE ULTIMATE AND TWO PENULTIMATES.

The ultimate (arc, A14; segment, A15) lies within the circle, *and the two penultimates* (intersections) bound it.

That is, part of one figure lies inside the other, and it has two boundaries.

A14 Theorem

SUTRA: IF THE SAMUCCAYE IS THE SAME THERE IS THAT WHICH IS ZERO.

If the samuccaye (insideness) is the same there is a zero (circle or other enclosing figure).

PART B

B1 Theorem

This is discussed along with the definition of magnitude, after Part D.

B2 Theorem

SUTRA: Paravartya yojayet, TRANSPOSE AND APPLY.

Flip the angle over and it fits its unflipped self (or apply A1 Theorem).

From here on a less formal presentation is sometimes used.

B3 Theorem

SUTRA: BY ADDITION AND SUBTRACTION

This Theorem and its converse come under the statement:

> the quality of an angle is unchanged by *addition*, and it is also unchanged *by subtraction.*

Note that, except that it applies to angles, this has some resemblance to the definition of magnitude.

B4 Theorem

Similar remarks apply here, too.

B5 Problem

SUTRA 1: TRANSPOSE AND APPLY.

For the construction is based on A2 Theorem, which comes under this sutra.

APPENDIX 1

However, symmetry is achieved by doubling up, which hints that we can usefully look for the sutra THE ULTIMATE AND TWO PENULTIMATES. For Example, *the ultimate* (a perpendicular, this being the objective) is specified by the intersection of *two penultimates* (two circles).

A third sutra which comes in is: IF THE SAMUCCAYE IS THE SAME THERE IS A ZERO. There is the phrase, 'to point the way'. It acknowledges that a point, which has *zero* magnitude, can be used to specify a direction from another point.
Given point P, a second point P' is sought such that the construction is *the same* on either side of line PP'. So we have here that which is zero, showing the presence of the sutra.

B6 Theorem

SUTRA: THE ULTIMATE AND THE PAIR OF PENULTIMATES.
The Theorem can be formulated thus: *the ultimate* (complete turn), *and the pair of penultimates* (half turns) taken together, are equal.

B7 Theorem

This is based on the Postulate. See 'Provision 4 and the sutras', page 111.

B8 Theorem

SUTRA: THE ULTIMATE AND THE PENULTIMATE PAIR.
See B6 Theorem and B5 Corollary.

B8 Corollary

SUTRA: THE ULTIMATE AND DOUBLE PENULTIMATE
The sutra is applied twice, to obtain a quadrupling.

B9 Theorem

SUTRA 1: Urdhva Tiryagbhyam, VERTICALLY AND CROSSWISE (Tirthaji's translation).
For opposite sides are equal, both *vertically and crosswise*.

SUTRA 2: TRANSPOSE AND APPLY
Turn a copy of the figure upside down and it fits the original. (See second proof).

APPENDIX 1

SUTRA 3: IF THE SAMUCCAYE IS THE SAME THERE IS THAT WHICH IS ZERO

For the first proof depends on the Postulate, which comes under this sutra.

PARALLELS

B10 Theorem

SUTRA 1: THE ULTIMATE AND TWO PENULTIMATES.

The Theorem may be restated incorporating the sutra:

> where a transversal cuts intersecting lines, corresponding angles between *the ultimate* (common direction) *and two penultimates* (two different directions) are unequal.

SUTRA 2: TRANSPOSE AND APPLY

One way of showing that two angles are unequal is to show that they cannot be constructed identically (see A2 Theorem).

B11 Theorem

SUTRA 1: THE ULTIMATE AND TWO PENULTIMATES

The Theorem may be so rephrased that it essentially incorporates the sutra:

> if corresponding angles with *the ultimate* (transversal) are equal, the *penultimate pair* of lines are parallel.

SUTRA 2: IF THE SAMUCCAYE IS THE SAME THERE IS THAT WHICH IS ZERO.

We are looking for some pair to be 'the same', and something which is zero. Rephrasing the Theorem to reflect this:

> if corresponding angles with the transversal are *the same*, the angle between the pair of lines is *zero*.

This implies that these lines are parallel – a result not demonstrated here, although it readily follows from Propositions B11 and B17, which latter could be placed before Proposition B11.

And this raises a question of general interest, concerning sutras: is it acceptable for a sutra to refer to something which has not been proved? I suggest that the answer is, 'Yes': theorems may be subject to a discipline of sequence, but there appears to be no reason why sutras should be.

APPENDIX 1

B12 Theorem

SUTRA: THE ULTIMATE AND TWO PENULTIMATES

Being the converse of B10 Theorem the application of the sutra follows much the same lines. Letting 'the ultimate' be a direction specified by the transversal, and 'the two penultimates' be directions away from the latter, specified by the intersecting pair of lines, we have:

> if corresponding angles between *the ultimate and two penultimates* are unequal, the pair of lines intersect.

B13 Theorem

This is the converse of B11 Theorem. The evidence so far suggests that the same sutras may well apply to theorems and their converses, and for the first sutra identified for B11 Theorem this is straightforward enough.

SUTRA 1: THE ULTIMATE AND TWO PENULTIMATES.

Rephrasing the Theorem to incorporate the sutra:

> corresponding angles with *the ultimate* (transversal) are equal, given that *the penultimate pair* of lines are parallel.

However, the second sutra given for B11 Theorem appears to be formulated the wrong way round for the present theorem:

SUTRA 2 (?): IF THE SAMUCCAYE IS THE SAME THERE IS THAT WHICH IS ZERO

In consequence we can choose between the following responses:

> 1. This sutra does not apply here.
> 2. The sutra does apply here, the principle being that if it applies to a theorem, and that theorem has a converse, it applies to the latter also.

The interesting option to explore is the second one. To get this to work, it is as though we need the converse sutra, which might be formulated as a subsutra:

IF THERE IS THAT WHICH IS ZERO, THERE IS THAT WHICH IS THE SAME,

or,

A ZERO INDICATES THE PRESENCE OF THAT WHICH IS THE SAME.

The zero refers to parallels here, as in B11 Theorem, so we can rephrase this subsutra:
IF THERE ARE PARALLEL LINES THERE IS THAT WHICH IS THE SAME,
or
PARALLELS INDICATE THE PRESENCE OF THAT WHICH IS THE SAME (corresponding angles) – which is a restatement of B13 Theorem.

However, are we entitled to turn a sutra round and use its converse? For without that the foregoing is invalid. The justification is that interpretation and application of the sutras is being done in a very free way, in order to explore the underlying hypothesis, that Tirthaji's sutras apply throughout mathematics. And in this spirit, the suggestion is that, where a sutra can have a converse, the latter comes under that sutra as a subsutra. In which case, it is like having a list (of two), in which the first item represents the whole list.

B14 Theorem
SUTRA 1: THE ULTIMATE AND TWO PENULTIMATES.
The Theorem can be reformulated thus, to incorporate the sutra:

> the ultimate (transversal) and *two penultimates* (parallel lines) make equal alternate angles.

SUTRA 2: IF THE SAMUCCAYE IS THE SAME THERE IS A ZERO.

Using the converse form, as in B13 Theorem, we have:

> parallel lines indicate the presence of that which is the same (alternate angles).

SUTRA 3: TRANSPOSE AND APPLY.
Rotate the figure and it fits its unrotated form.

SUTRA 4: VERTICALLY AND CROSSWISE.

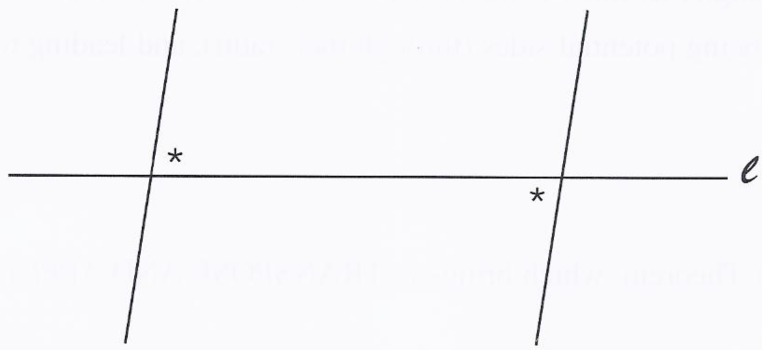

Fig 43

The *vertically* opposite forms are congruent (indistinguishable, above and below line ℓ) *and crosswise* the angles are equal. Note that the first phrase refers to what amounts to another theorem. That is, this is a further example of two theorems being covered by one sutra.

B15 Theorem

SUTRA 1: THE ULTIMATE AND TWO PENULTIMATES.

Rephrasing the Theorem:

> if alternate angles between *the ultimate* (transversal) *and two penultimates* (straight lines) are equal, those straight lines are parallel.

SUTRA 2: IF THE SAMUCCAYE IS THE SAME THERE IS THAT WHICH IS ZERO.

That is, if alternate angles are equal, the angle between the lines intersected by the transversal is zero, showing that the latter are parallel.

SUTRA 3: TRANSPOSE AND APPLY

Turning the figure round, the parts of the figure above and below line ℓ are congruent (Fig 43).

SUTRA 4: VERTICALLY AND CROSSWISE.

It is the same story as in B14 Theorem: *vertically* (above and below line ℓ) the figures are congruent, *and crosswise* the angles are equal.

TRIANGLES

B16 Problem

SUTRA: THE ULTIMATE AND TWO PENULTIMATES

The objective is the triangle; let this be 'the ultimate'. The two circles used in the construction are then penultimates, being potential sides (through their radii), and leading to completion of the triangle.

B16 Corollary

The proof draws on A1 Theorem, which brings in TRANSPOSE AND APPLY.

APPENDIX 1

B17 Theorem

SUTRA: BY COMPLETION OR NON-COMPLETION.

The half turn is completed by summing the angles of the triangle.

B18 Theorem

SUTRA: BY COMPLETION OR NON-COMPLETION.

Summing the interior opposite angles does not complete the half turn, but it does complete (equal) the exterior angle.

SUTRA 2: Yavadunam, WHATEVER THE DEFICIENCY (Tirthaji's translation).

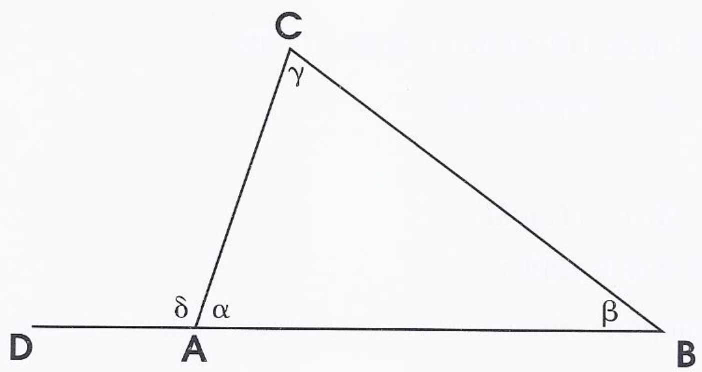

Fig 44

Whatever the deficiency (α) its complements are equal, these being $\beta + \gamma$ by B17 Theorem, and δ.

The Theorem follows

SUTRA 3: TRANSPOSE AND APPLY.

Transpose angles B and C to vertex A, using the theory of parallels. *And apply* the definition of a half turn, when the result follows.

B19 Theorem

SUTRA: TRANSPOSE AND APPLY.

Turning the triangle over, it is unchanged.

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B20 Theorem

SUTRA: THE ULTIMATE AND TWICE THE PENULTIMATE.

The theorem can be reformulated thus: the ultimate (exterior angle) and twice the penultimate (a base angle) are equal.

The exterior angle is regarded as 'the ultimate', here, because it is what we are interested in. And the penultimates are the base angles, which together (or one of them doubled) give the ultimate. It is in this sense that the penultimates are next to the ultimate. This example shows the freedom with which the terms 'ultimate' and 'penultimate' may be applied.

QUADRILATERALS

B21 Theorem

SUTRA: BY COMPLETION OR NON-COMPLETION.

Together, the angles form a complete turn.

B22 Problem and Corollaries (1) and (2)

SUTRA: TRANSPOSE AND APPLY

The proofs here draw on propositions A1 and A2, which come under this sutra.

B23 Theorem

SUTRA 1: TRANSPOSE AND APPLY.

This seems to be the principle sutra here, since all four angles can be constructed identically.

However, other sutras are relevant, and can be unearthed by looking at the Theorems referred to in the proof:

SUTRA 2: BY COMPLETION OR NON-COMPLETION (From B21 Theorem)

SUTRA 3: THE ULTIMATE AND TWICE THE PENULTIMATE (From B6 Theorem)

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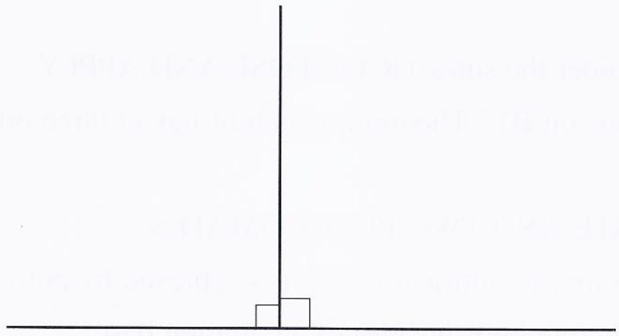

Fig 45 Right angles

Take a perpendicular to a straight line, and the sutra applies: the half turn (ultimate) and two equal angles (right angles = penultimates) are equal (Fig 45).

The same applies on the other side of the line, and one line intersecting another at right angles does both jobs at once (Figure 46). In the process a complete turn is divided into four equal angles, or right angles.

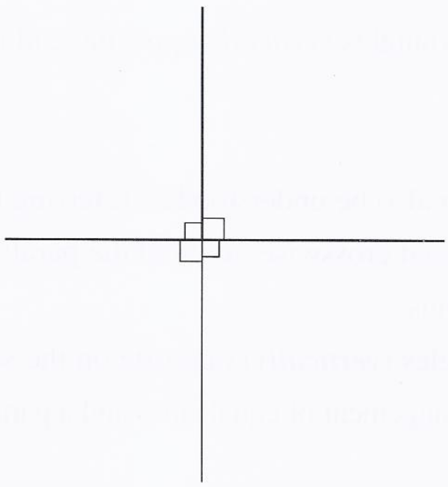

Fig 46 Two pairs of right angles

B24 Theorem

SUTRA: TRANSPOSE AND APPLY.

The equalities of sides are established using this sutra. For the proof depends on B22 Corollary (2), which in turn depends on A1 Theorem, which comes under this sutra.

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B25 Problem

Parallels are established under the sutra TRANSPOSE AND APPLY.

However, the proof depends on B15 Theorem, which brings in three other sutras.

SUTRA 2: THE ULTIMATE AND TWO PENULTIMATES

The goal is the parallelogram (the ultimate), and it is achieved by putting two congruent triangles (the penultimates) on opposite sides of a common base.

SUTRA 3: IF THE SAMUCCAYE IS THE SAME THERE IS THAT WHICH IS ZERO.

'The Samuccaye is the same' can refer to the two congruent triangles, and the zero to either pair of parallel lines.

SUTRA 4: VERTICALLY AND CROSSWISE.

'Crosswise' can refer to the pairs of parallel lines, and 'vertically' to the oppositely placed congruent triangles (B25 Corollary 1).

We can then say,

> place two congruent triangles *vertically* opposite, and there are two pairs of parallels *crosswise*.

However, the 'crosswise' can also be understood as referring to equal sides of the congruent triangles needing to be arranged crosswise, to yield the parallelogram. With this interpretation we can restate the theorem thus:

> place congruent triangles (*vertically*) opposite on the same base, so that there is a *crosswise* arrangement of equal sides and a parallelogram results.

B25 Corollaries (1) and (2)

Some of the sutras relevant to B25 Problem apply to these corollaries also.

B26 Theorem

The proof given refers to three theorems: B13, B8 and B25 Corollary (2), which see for discussion of the relevant sutras.

These are:

SUTRA 1: IF THE SAMUCCAYE IS THE SAME THERE IS A ZERO (converse form)

APPENDIX 1

SUTRA 2: THE ULTIMATE AND TWO PENULTIMATES.

SUTRA 3: TRANSPOSE AND APPLY.

PART C

Proposition C2, that area is a magnitude, is covered under the discussion on magnitude on page 111. The Definition of area, and the rest of Part C, come under the sutra BY ADDITION AND SUBTRACTION.

That other sutras also come in can be seen by considering the theorems referred to in the proofs, and the sutras applying to them.

PART D

We owe our use of a circle to represent zero to the Arabs, who got it from the Hindus. Its origin may predate writing, and it may be part of a symbolism in which 'zero' and 'a circle' symbolise each other: such symmetry is the sort of thing mathematicians generally appreciate. Be that as it may, in what follows it will be assumed that *zero symbolises a circle,* as well as retaining its ordinary meaning.

The sutra, IF THE SAMUCCAYE IS THE SAME THERE IS A ZERO, is represented in a number of the following examples in its converse form:
IF THERE IS A ZERO (circle) THERE IS THAT WHICH IS THE SAME. [E.g. a given chord of a circle subtends equal angles on the (same arc of the) circle]. Propositions it applies to include: D2, D3, D4, D5, D8, D9 AND D10.

The following study shows which other sutras come into play.

D1 Theorem
SUTRA: BY COMPLETION OR NON-COMPLETION.
The complete figure establishes an inscribed rectangle; the incomplete figure relates a right angle to the diameter (= a diagonal).

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D2 Theorem

SUTRA: BY COMPLETION OR NON-COMPLETION.

Observation establishes that opposite angles of a cyclic-quadrilateral total a half turn.

D3 Theorem

The proof shows two relevant sutras:

BY COMPLETION OR NON-COMPLETION applies via D2 Theorem;

BY ADDITION AND SUBTRACTION shows up in the first three steps of the proof.

D4 Theorem

Again, BY ADDITION AND SUBTRACTION can be seen at work in the proof. Additional sutras are brought in by the theorems referred to in the proof.

D5 Theorem

SUTRA: chalanakalanabhyam, BY A LIMITING PROCESS.

Tirthaji does not give an English translation of this sutra. Its application here is self-explanatory.

D6 Theorem

SUTRA: THE ULTIMATE AND TWICE THE PENULTIMATE.

This is also self-explanatory.

D7 Theorem

SUTRA: Anurupye Shunyamnyat, IF ONE IS IN RATIO THE OTHER ONE IS ZERO.

Here, (i) the phrase 'in ratio' is taken to refer to similar triangles;

(ii) working with the converse of the sutra, as a subsutra (see notes on B13 Theorem), we have:

IF ONE IS ZERO, THE OTHER ONE IS IN RATIO;

(iii) zero is used to symbolize a circle.

The Theorem can accordingly be reformulated so as to reflect the subsutra:

if one part of the figure is a circle (zero) the other part is a pair of similar triangles (in ratio).

D8 Theorem, D9 Theorem and D10 Theorem

SUTRA: IF ONE IS IN RATIO THE OTHER ONE IS ZERO.

Again, 'in ratio' refers to similar triangles, and the converse form (subsutra) is used, and zero symbolises a circle. The 'equal areas' property follows, given similar triangles.

APPENDIX 1

The definition of magnitude and Propositions A11, B1 and C2

1. This definition deals with the *whole* and the *part*. The closest to this among the sutras is,

BY COMPLETION OR NON-COMPLETION.

Whatever is *complete* is *whole* and whatever is not is *partial*. Completion is the act of making complete. But why *by* completion? It seems to refer to something in the act of happening, which is a very alive quality. Is it telling us that we need to look at what it is that makes something whole?

One of the interesting things about a magnitude is that the complement needed to make a part the same as the whole is itself a magnitude. The whole has been split into two parts, and both are magnitudes. The definition only refers to one of them, apparently, but *it could be either part*. That is, *the definition applies equally to both parts*. The part required BY COMPLETION and the part currently in a state of NON-COMPLETION share a common quality, such that they can be brought together to form a larger whole, which also has that quality.

2. Secondly, as already hinted at, the sutra BY ADDITION AND BY SUBTRACTION applies. For, whether adding or subtracting, what we have remains a magnitude.

Provision 4 and the sutras

First, a few points to clear the way:

(i) One meaning of *zero* is *absence of magnitude*.

(ii) Motion can be indicated by one or more magnitudes, such as velocity and acceleration. When these are zero, there is rest.

(iii) Since zero can indicate the presence of rest, for the purposes of the sutras we can take as one of its meanings that ZERO = REST.

(iv) Furthermore, the principle that all motion is relative also has implications concerning the state of rest, for it follows that *there is rest within every moving system*. This is a corollary of the principle that motion is relative.

(v) These points can now be harnessed to restate Provision 4 in a form paralleled by one of the suggested sub-sutras:

there being rest in every state of motion, there are things which remain constant (in this case *magnitudes*) whatever the state of motion.

This can be expressed more tersely:

IF THERE IS A ZERO (REST) THE MAGNITUDE REMAINS THE SAME (WHATEVER THE STATE OF MOTION).

Or as the suggested sub-sutra puts it:

IF THERE IS ZERO, THE SAMUCCAYE REMAINS THE SAME.

CONCLUSIONS

Flexible interpretation and application of the sutras makes it possible to apply them to all the propositions given here. The hypothesis that the sutras apply to each and every part of mathematics is supported by this study. They seem to be an attempt to sum up verbally underlying patterns, themes and principles which keep cropping up. Words and phrases can have different meanings, and this seems to apply especially to the sutras. Are these the *sound-bites* of mathematics?

The three main conclusions are:

1. Alternative translations of the sutras are needed in some cases.
2. In a couple of cases there is a need for the converse of a sutra. This is assigned the status of a subsutra coming under that sutra.
3. Zero symbolises a circle, as well as retaining its ordinary meaning.

At more length, the features of these conclusions are:

1. Tirthaji's English translations of the sutras do not always convey their full meanings, so that some alternatives are needed.

APPENDIX 1

Alternative translations proposed:

(i) Sutra: Shunyam samyasamuccaye. *Tirthaji's translation:* if the samuccaye is the same that samuccaye is zero.

Alternative translation: if the samuccaye is the same there is that which is zero.

(ii) Sutra: sopantyadvayamantyam.

Tirthaji's translation: the ultimate and twice the penultimate.

Alternative translation: the ultimate and double penultimate.

This last being ambiguous, one meaning is conveyed by Tirthaji's translation and another by: the ultimate and two penultimates.

2. In two cases it is assumed that the sutra also stands for its converse, these being given here as subsutras.

The examples are:

(i) Sutra: if the samuccaye is the same there is that which is zero.

Subsutra (converse): if there is a zero there is that which is the same.

(ii) Sutra: if one is in ratio the other one is zero.

Subsutra (converse): if one is zero the other one is in ratio.

3. Zero, or 'that which is zero', can:

(i) symbolise a circle,

(ii) indicate a point (which has zero magnitude),

(iii) indicate a *direction* from a point, since a second point suffices for this,

(iv) represent parallels, the angle between them being zero,

(v) denote the absence of a magnitude, and hence it can denote rest.

Other features are:

4. In the sutra, 'the ultimate and twice the penultimate', the 'ultimate', being some aim or goal, may be freely chosen. What is considered to be 'penultimate' is in some way next to it.

5. It is assumed that the phrase 'one is in ratio' can be a reference to similar triangles, in the sutra 'if one is in ratio the other one is zero', and its converse.

6. Examples have been given in which two theorems jointly come under one sutra.

7. It is suggested that a sutra may refer to something which has not yet been proved.

8. Tirthaji does not give a translation for the sutra 'chalanakalanabhyam'. The translation suggested here is, 'by a limiting process'.

However, on page 185 of *Vedic Mathematics* Tirthaji gives, 'Chalana-Kalana – Differential Calculus'. Is this intended to be a translation?

SUBSUTRAS

As well as the sixteen sutras Tirthaji uses a number of subsutras. An example is 'chakravat', cyclically. For example, the angles of a triangle or quadrilateral may conveniently be summed in cyclical sequence. Not being essential to this study, subsutras have been omitted – a couple of converses excepted.

According to Somanath Mahapatra, who was taught vedic mathematics by Tirthaji, 'he used to pluck them ((subsutras)) from the air'– the above being an example. The other examples he was given are to be found in the book *Vedic Mathematics,* and also in his own book, *Vedic Ganit* (written in the Oriian language). Mahapatra was also given a list of the sixteen sutras by Tirthaji, confirming the list in *Vedic Mathematics*, which was compiled from stray references in the text.
[The author met Somanath Mahapatra in Puri, India, in 1981].

OBSERVATIONS

(i) Sometimes a sutra shows up in the statement of a theorem, sometimes in the proof, and sometimes sutras cover both – although not necessarily the same ones.
(ii) The sutra does not always reveal itself at first sight.

CONCLUDING REMARKS

Mathematicians are constantly looking for pattern. If the sixteen sutras really do refer to themes or patterns running throughout the whole of mathematics, to have formulated them is a fine achievement – and a valuable contribution.

APPENDIX 1

QUESTIONS, POSSIBLY ANSWERS

1. Tirthaji evidently had in mind that the sixteen sutras apply to his system. But to what extent do they apply to mathematics not fitting an oral tradition? Might it be that the sutras still apply, but are scattered more thinly?

2. What is meant by the suggestion that zero symbolises a circle?
If A can symbolise B, why cannot B symbolise A? There is no need to know what this means in order to suggest it: the meaning is something which can be explored. As already remarked, the need here is to take a fresh look at things to free up the thinking. It is conceivable that the word 'zero', even if not the concept, might be used to symbolise a circle. It is conceivable that a society or people might regard symbolism to be reversible, what is symbolised and the symbol switching roles. In this connection note the following extract from what the *New Shorter Oxford Dictionary* has to say about the word 'symbol':

> a thing conventionally regarded as representing, typifying, or recalling something else by possessing analogous qualities or by association in fact or thought.

What analogous qualities do zero and a circle possess? Symmetry is one. For no figure is more symmetrical than a circle – with the possible exception of a point – and can anything be more symmetrical than zero, the absence of anything?

APPENDIX 2

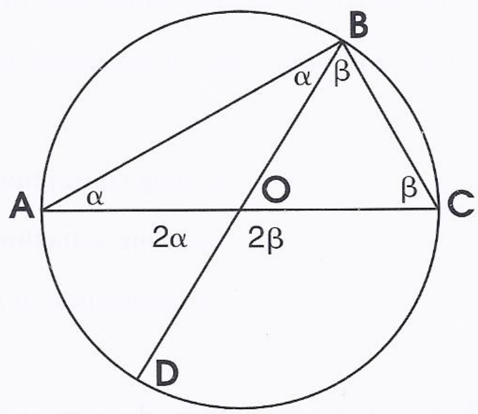
ALTERNATIVE PROOFS & SEQUENCES IN PART D

Two alternative sequences are given, which between them cover Propositions D1 - D5, plus one extra theorem, that a diameter is perpendicular to the tangent through an end point. These sequences are used to present alternative proofs.

SEQUENCE 1

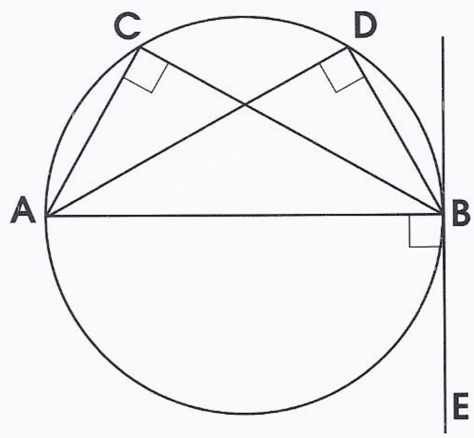
Theorem 1 A diameter subtends a right angle on the circumference of the circle.

In the figure, O being the centre of the circle,
triangles OAB and OBC are isosceles.
Therefore,
angle DOA = twice angle DBA [B20 Theorem]
Likewise,
angle DOC = twice angle DBC [B20 Theorem]
And AOC being a diameter, together these make a half turn.
Therefore angle ABC = angle DBA + angle DBC is half of this, i.e. a right angle, as required.



Theorem 2 A diameter to the circle is perpendicular to the tangent at its end point.

In the figure, AB being a diameter subtends a right angle on the circumference. [Theorem 1]
Let the chord DB get shorter and shorter.
In the limit the points D and B coincide, so that BE is a tangent, but it remains the case that angle ABE is a right angle.This establishes the Theorem.



APPENDIX 2

Theorem 3 The angle between the tangent and the chord equals the angle subtended by that chord on the far arc.

CASE I (Point O lies inside ΔBCD)

In the figure, AB is a tangent at B and O is the centre of the circle.

In ΔBDC, $2\alpha + 2\beta + 2\gamma = \frac{1}{2}$ turn.

Therefore $\alpha + \beta + \gamma = 1$ right angle.

But angle ABO = 1 right angle. [Theorem 2]

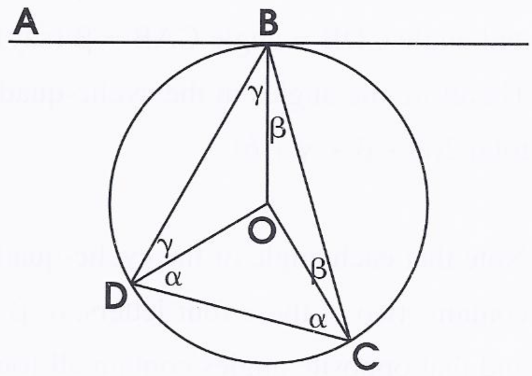
Therefore angle ABO = angle ABD + angle DBO

$= (\alpha + \beta) + \gamma$.

That is, angle ABD $= \alpha + \beta$

= angle BCD,

demonstrating the Theorem in this case.



CASE II (Point O lies outside ΔBCD)

In ΔBCD, $\alpha + \gamma + (\gamma - \beta) + (\alpha - \beta) = \frac{1}{2}$ turn.

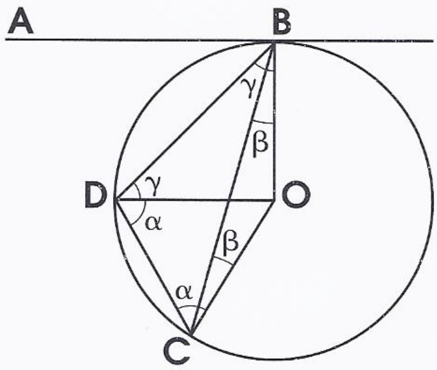
Therefore $\alpha + \gamma - \beta = 1$ right angle.

But angle ABO is a right angle,

and a part of it, angle DBO $= \gamma$.

Therefore angle ABD $= \alpha - \beta$ = angle DCB,

demonstrating the Theorem in this case also.

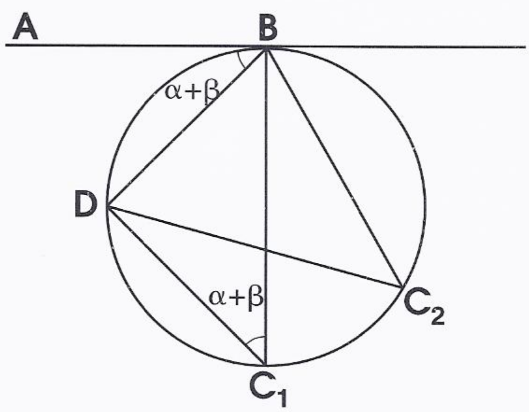


Theorem 4 Equal chords subtend equal angles on the same arc.

For by Theorem 3,

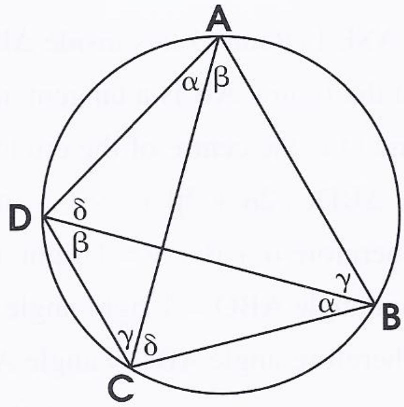
angle ABD = angle BC_1D = angle BC_2D.

This proves the Theorem.



APPENDIX 2

Theorem 5 Opposite angles of a cyclic-quadrilateral total a half turn.



In the figure, by Theorem 4,
angle DAC = angle DBC = α (say)
and angle CDB = angle CAB = β (say), etc.
Therefore the angles in the cyclic-quadrilateral total $2(\alpha + \beta + \gamma + \delta)$.

Note that each angle of the cyclic-quadrilateral contains two of these four letters, α, β, γ and δ, and that opposite angles contain all four of them.
Therefore angle A + angle C = $\alpha + \beta + \gamma + \delta$ = angle B + angle D.
But angle A + angle C + angle B + angle D = 1 turn.
Therefore angle A + angle C = ½ turn,
demonstrating the Theorem.

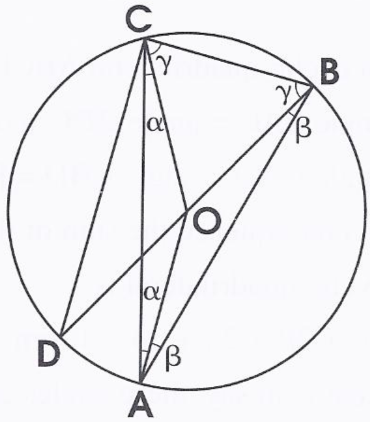
SEQUENCE 2

Theorem 1 A diameter subtends a right angle on the circumference of the circle.
Use the proof in Theorem 1, Sequence 1 (or D1 Theorem)

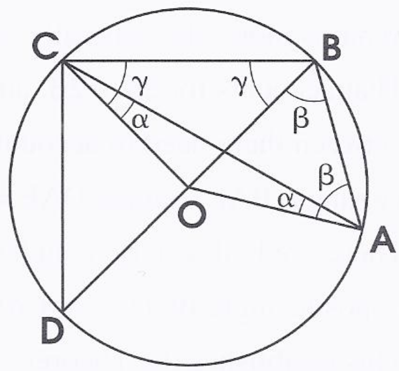
APPENDIX 2

Theorem 2 Equal arcs subtend equal angles on the circumference.

CASE I (Point O lies inside ΔABC)
In ΔABC, $2\alpha + 2\beta + 2\gamma = \frac{1}{2}$ turn.
Therefore $\alpha + \beta + \gamma = 1$ right angle.
But angle DCB = 1 right angle.
And it is split into three angles, α, γ and angle DCA, where
angle ACO = α,
and angle OCB = γ.
Therefore angle DCA = β = angle DBA.
That is, arc DA subtends α constant angle β on the circumference, so long as Fig 1 applies.



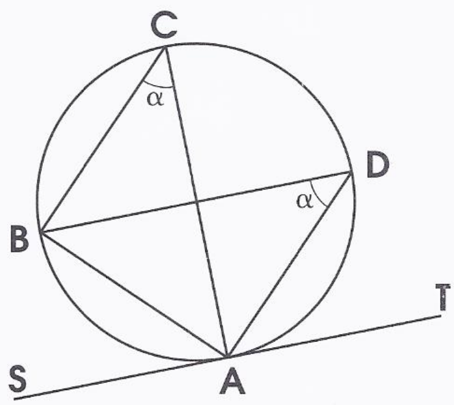
CASE II (Point O lies outside ΔABC)
In ΔABC,
$\gamma + \beta + (\beta - \alpha) + (\gamma - \alpha) = \frac{1}{2}$ turn.
Therefore $\gamma + \beta - \alpha = 1$ right angle.
But angle DCB = 1 right angle [Theorem1]
and angle OCB = γ.
Therefore angle DCO = $\beta - \alpha$.
Therefore angle DCA = β = angle DBA.



That is, arc DA subtends a constant angle on the circumference, so long as Fig 2 applies.
By A2 Theorem, this applies to any identical construction using an equal arc AD.
Therefore the Theorem is established in both cases.

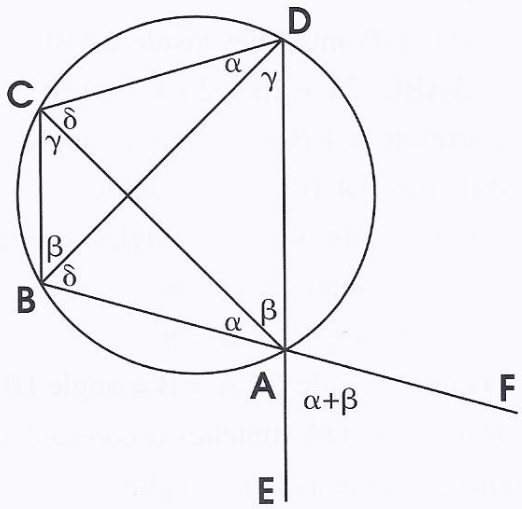
Theorem 3 The angle between the tangent and the chord equals the angle subtended by that chord on the far arc.

The angles at C and D being the same [Theorem 2],
suppose that positions of D are taken
closer and closer to A.
In the limit, DA is replaced by the tangent TA,
and angle BDA becomes angle BAS.
That is, angle BAS = angle BCA, as the Theorem states.



APPENDIX 2

Theorem 4 An exterior angle of a cyclic-quadrilateral equals the interior opposite angle.



In cyclic-quadrilateral ABCD, by Theorem 3,
angle BAC = angle BDC = α (say),
angle CAD = angle CBD = β (say), etc.
In consequence the sum of the angles of the cyclic-quadrilateral is
$2\alpha + 2\beta + 2\gamma + 2\delta = 1$ turn. [Prop. B21]
That is to say, these angles can neatly be arranged around a point – such as point A.
And at A we have a start, for angle BAD = $\alpha + \beta$.
What is more, the vertically opposite angle must also equal $\alpha + \beta$, i.e. angle EAF = $\alpha + \beta$.
That accounts for $2\alpha + 2\beta$, and there are two remaining vertically opposite angles at A, which between them need to account for the remaining $2\gamma + 2\delta$.
i.e. angle BAE = angle DAF = $\gamma + \delta$.
These are both external angles of the cyclic-quadrilateral, and they are equal to the interior opposite angle BCD (= $\gamma + \delta$).
This establishes the Theorem.

Also good is Euclid's sequence in the *Elements*, Book III, Propositions 20, 21, 22.

APPENDIX 3

FURTHER DEFINITIONS

The following definitions are offered as a basis for further discussion.

1. Tangent

This definition is given in negative form: by excluding what it is not, what remains is what it is.

Definition If a straight line meets a line which is not straight, and if neither line is terminated and they do not cross at that meeting point, and do not share a stretch of line, then the straight line is said to be a *tangent* to the other line at that point.

2. Extension to three dimensions

Definition All those points included when a plane rotates about a straight line on it are said to belong to *space*.

Definition A collection of surfaces and/or lines and/or points is called a *figure*.

Definition And if the figure lies purely in a plane let it be called *a figure in a plane* or *plane figure*.

Concerning figures in space

Definition If two points can be linked without crossing a figure they are said to lie on the *same side* of that figure, and otherwise on *different sides*.

Definition If a finite figure has just one side then it is said to be *open.*

Definition If a finite figure has two or more sides then it is said to be *closed.*

Definition A point is said to lie *outside* a closed figure if it can be linked to a straight line without either line intersecting that figure.

Definition If a straight line not in a given plane but standing on a point which is, makes the same angle with all straight lines within the plane and which stand on that point, then it is said to be *normal* to that *plane*, and the plane is said to be normal to that line.

APPENDIX 3

Definition That whose elements are surfaces, lines and points, and which at each point embodies a complete turn normal to all directions in space, is called a *solid*.

The case is similar to that of a surface. A boundary to a solid does not satisfy the above definition of a solid, and the solid may contain holes or slits.

Note that, as defined, a solid need not be of finite extent. But, if it is, and if it contains a single hole running right through, then its shape is essentially that of a torus, or doughnut.

The following notation is intended to be of use where there is a need to distinguish between an acute and an obtuse angle.

As an example, suppose we are given a cyclic-quadrilateral ABCD, O being the centre of the circumscribing circle. The two angles AOC can be distinguished by the notation: angle BAOCB for the one and angle DAOCD for the other. Of course, if an angle is known to be acute, three letters suffice for it.